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THE GEOMETRY OF THE ASYMPTOTICS OF POLYNOMIAL MAPS

BY

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ABSTRACT

The aim of this paper is to develop a theory for the asymptotic behavior of polynomials and of polynomial maps over R and over C and to apply it to the Jacobian conjecture. This theory gives a unified frame for some results on polynomial maps that were not related before.

A well known theorem of J. Hadamard gives a necessary and sufficient condition on a local diffeomorphism $f\colon R^n \to R^n$ to be a global diffeomorphism. In order to show that f is a global diffeomorphism it suffices to exclude the existence of asymptotic values for f.

The real Jacobian conjecture was shown to be false by S. Pinchuk. Our first application is to understand his construction within the general theory of asymptotic values of polynomial maps and prove that there is no such counterexample for the Jacobian conjecture over C. In a second application we reprove a theorem of Jeffrey Lang which gives an equivalent formulation of the Jacobian conjecture in terms of Newton polygons. This generalizes a result of Abhyankar. A third application is another equivalent formulation of the Jacobian conjecture in terms of finiteness of certain polynomial rings within $C[U, V]$.

The theory has a geometrical aspect: we define and develop the theory of etale exotic surfaces. The simplest such surface corresponds to Pinchuk's construction in the real case. In fact, we prove one more equivalent formulation of the Jacobian conjecture using etale exotic surfaces. We consider polynomial vector fields on etale exotic surfaces and explore their properties in relation to the Jacobian conjecture.

In another application we give the structure of the real variety of the asymptotic values of a polynomial map $f\colon R^2 \to R^2$.

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1. Introduction

The aim of this paper is to develop a theory for the asymptotic behaviour of polynomials and of polynomial maps over R and over C.

If $f\colon K^n \to K^n$ is a map (K = R or C) then a point $X_0 \in K^n$ is called an **asymptotic value** of f if there exists a curve $\sigma(t) = (X_1(t), \dots, X_n(t))$, $0 \le t < \infty$, which extends to ∞ so that

$$\lim_{t\to\infty} f(\sigma(t)) = X_0;$$

$\sigma(t)$ is called an **asymptotic curve** of f.

If $f = (f_1, \dots, f_n)$ and $X_0 = (X_{01}, \dots, X_{0n})$, then we say that X_{0j} is an asymptotic value of the function f_j, $j = 1, \dots, n$.

A well known theorem of J. Hadamard [4] gives a necessary and sufficient condition on a local diffeomorphism $f\colon R^n \to R^n$ to be a global diffeomorphism (Section 2). As a consequence one deduces that in order to show that f is a global diffeomorphism it suffices to exclude the existence of asymptotic values for f. Hence the relevance of the study of asymptotic values of polynomials and polynomial maps to problems of the type of the Jacobian conjecture [2], [17].

Let $F\colon K^n \to K^n$ be a polynomial map (K = R or C). Let us denote by $J(F)$ the determinant of the Jacobian of F. The celebrated Jacobian conjecture is the following statement: *If $J(F)$ never vanishes then the map F is injective.*

Indeed, originally, the problem was stated over $K = C$ (even $K = Z$) and was conjectured by O. Keller [5]. In this setting the assumption can be rewritten as $J(F) \in C^*$ (by the Fundamental Theorem of Algebra) and the conclusion can be rewritten as: *F is invertible in the ring $C[X_1, \ldots, X_n]$* [3].

For $K = R$ it is known that any injective polynomial map F: $R^n \to R^n$ is surjective [6], [16]. This result was generalized by A. Borel to real algebraic varieties.

The conjecture for $K = C$ is still open for $n \geq 2$. We shall concentrate on the case $n = 2$.

However, recently the real conjecture was shown to be false by S. Pinchuk [13], [10], [11], [12]. His beautiful counterexample is of a polynomial map (P, Q) which consists of two polynomials in the ring $R[t, h, f]$ where

$$t = XY - 1, \quad h = t(Xt + 1), \quad f = (Xt + 1)^2(h + 1)/X.$$

If we define

$$u_1 = Y, \quad u_2 = XY, \quad u_3 = X^2Y - X,$$

then one easily checks that, in fact, Pinchuk's polynomials belong to the ring $R[u_1, u_2, u_3]$.

This ring is a special one. It arises naturally in our theory of asymptotics. To explain how, let us define the following birational map:

$$R(X, Y) = (X^{-1}, X + X^2Y).$$

Then the ring $R[u_1, u_2, u_3]$ consists of all the polynomials $P(X, Y) \in R[X, Y]$ with the property

$$(P \circ R)(X, Y) = P(X^{-1}, X + X^2Y) = A(X, Y) \in R[X, Y]$$

(Proposition 6, Section 6). We denote this ring by $I(R(X, Y))$. We say that the polynomials in $I(R)$ satisfy **an asymptotic identity with respect to the rational map** $R(X, Y)$.

If (P, Q) is any pair of polynomials in $I(R)$, then the map they induce satisfies the following **double asymptotic identity**:

$$P(X^{-1}, X + X^2Y) = A(X, Y) \in R[X, Y],$$
$$Q(X^{-1}, X + X^2Y) = B(X, Y) \in R[X, Y].$$

Thus $\lim_{X \to 0}(P, Q)(X^{-1}, X + X^2Y) = (A(0, Y), B(0, Y))$, so that the points of the affine algebraic curve $(A(0, Y), B(0, Y))$ are all asymptotic values of the map

(P, Q). As a consequence of this and of the Theorem of Hadamard, in order to construct a counterexample to the real Jacobian conjecture it suffices to find a Jacobian pair within $I(R(X, Y))$. This was demonstrated in Pinchuk's work.

It might be tempting to look for a counterexample for Keller's problem ($K = C$) in the ring $I(R(X, Y))$. Any such example, if it exists, should be of a high degree (at least 100, see [9]). However, one consequence of our asymptotic theory is the following (Theorem 17, Section 8): *There is no counterexample to the complex Jacobian conjecture with coordinates in* $C[V, VU, VU^2 + U]$ (see [11]).

In fact, there is no such counterexample in a much wider family of rings. Thus one cannot hope to find a counterexample to Keller's problem which is of the type constructed by Pinchuk.

The usefulness of the theory of asymptotics lies in that it puts under one frame results which are algebraic and results which are algebro-geometric. Let us mention a few instances of this.

A well known theorem due to Abhyankar [1] asserts the following: *Every Jacobian pair is an automorphic pair iff the Newton polygons of every Jacobian pair are triangles with vertices on the coordinate axes.*

Jeffrey Lang [7] proved a stronger statement, namely: *Every Jacobian pair is an automorphic pair iff the Jacobian condition implies that the Newton polygons have no edges of a positive slope.*

We prove this last theorem of Lang (Section 4) by analysing the simple relations between the Newton polygon and Walker's boundary of a polynomial. This last device is essential in our theory of asymptotics to derive the various types of asymptotic identities in the complex case (see [18], page 98 and Section 3).

As another algebraic application we give (Section 6) the following equivalent formulation to the Jacobian conjecture (Theorem 12, Section 7): *The complex Jacobian conjecture is true iff any Jacobian pair* $P(U, V), Q(U, V) \in C[U, V]$ *induces a finite map* $f = (P, Q)$. Indeed, this is merely a simple conclusion of the fact that the subalgebra $I(R)$ of $C[U, V]$ is not finite over $C[U, V]$ (Definition 9, Section 7) for any rational map $R(X, Y)$ which is not polynomial (Theorem 11, Section 7).

As for the geometrical aspects of this theory, these are related to the theory of exotic surfaces [12]. We are naturally led to define a new type of exoticity (stronger than the standard one). We say that a surface (affine) S is etale exotic if:

(a) There is a diffeomorphism ϕ: $C^2 \to S$ which is realized by a birational map ϕ.

(b) There is no regular etale map $S \to C^2$ (into C^2).

An example of such a surface is the surface $S_3 \subseteq C^3$ that is parametrized by

$$X = V, \quad Y = VU, \quad Z = VU^2 + U.$$

Note that these are exactly the generators u_1, u_2, u_3 of $I(X^{-1}, X^2Y - X)$ from where Pinchuk extracted his polynomials. The affine closure of S_3 is $XZ = Y(Y+1)$. It is rather standard to show that S_3 is not isomorphic to C^2 (Theorem 13, Section 8).

In fact, if we regard S_3 to be a real surface (in R^3), then it cannot be mapped diffeomorphically onto R^2 by a polynomial map (Theorem 14, Section 8). However, according to Pinchuk there are regular etale maps $S_3 \to R^2$ (Theorem 15, Section 8).

As mentioned above, the situation is completely different for the complex S_3 (since it is etale exotic). This is a special case ($N = 2$) of Theorem 16 (Section 8), that provides us with a whole family, S_N, of etale exotic surfaces. S_N is embedded in C^N.

The proof of that theorem is purely algebraic and long. If one could prove such a theorem for all of the surfaces S_R (see the definition of S_R prior to Proposition 9, Section 8) then this would prove the validity of the complex Jacobian conjecture.

Using a theorem of H. Kraft that identifies the space $\mathrm{SL}_2(C)/C^*$ with the affine closure of S_3, $XZ = Y(Y+1)$, we can deduce that there are no regular etale maps $\mathrm{SL}_2(C)/C^* \to C^2$.

Finally, we mention that some innocent looking surfaces such as the complex sphere $X^2 + Y^2 + Z^2 = 1$ cannot be regularly etale mapped into C^2 (Theorem 22, Corollary 5, Section 8).

In Section 9 we consider some polynomial vector fields on etale exotic surfaces. We explain the following phenomenon: *For any two polynomials $P(X,Y,Z)$, $Q(X,Y,Z) \in C[X,Y,Z]$ the vector field $\nabla P \times \nabla Q$ (the vector product of the gradients) on S_3 always contains vectors which lie in the tangent plane to S_3 at the point of evaluation.*

This is merely a special case of Theorem 24, Section 9. We explain (Proposition 13, Section 9) how the standard proofs of such properties with respect to smooth functions are based on the Stokes Theorem. However, the Stokes Theorem requires compactness of the surface, which is certainely not the case in our algebraic surfaces. On the other hand, we need the conclusion only for polynomial fields (and not just any smooth fields). Thus maybe there is some version of an Algebraic Stokes Theorem.

All the above motivates study of the asymptotics of polynomials and of polynomial maps. We study the structure of asymptotic identities in Section 3, for $K = \mathrm{R}$, and in Section 4, for $K = C$. We give there a few theorems — algebraic in nature — which have some interest on their own. For example, we give the structure of the real variety of the asymptotic values of a polynomial map $f\colon R^2 \to R^2$ (Theorem 4, Section 3) (see [10]).

We hope that the results presented in this paper will motivate further research on the asymptotics of polynomials.

2. A theorem of J. Hadamard and asymptotic values

We refer to the paper by J. Hadamard [4]. A copy of this paper appears also in J. Hadamard Selecta (145–159).

Let $f\colon R^n \to R^n$ be a C^1 map such that for any $x \in R^n$, $\det J(f)(x) \neq 0$. For each $x \in R^n$ we let

$$N(x) = 1/ \parallel D_f^{-1}(x) \parallel;$$

D_f is the differential of f and D_f^{-1} is its inverse operator. For any $R > 0$ we denote

$$\mu(R) = \min_{\|x\|=R} N(x).$$

THEOREM (J. Hadamard, [4]): *If f satisfies*

$$\int_0^\infty \mu(R)dR = \infty \tag{1}$$

then f is a global diffeomorphism of R^n onto R^n.

Since the integrand in (1) gives a lower bound to the length dilatation at distance R from the origin, and since by continuity of the dilatation (in our case) there are curves that extend to ∞ that attain this bound as close as we please, we can restate condition (1) as follows:

The image of any curve

$$\sigma(t), \quad 0 \leq t < \infty, \tag{2}$$

which extends to ∞ under f, i.e. $f(\sigma(t))$, $0 \leq t < \infty$ is a non-rectifiable curve.

Definition 1: Let $f\colon R^n \to R^n$ be any smooth map. A point $x_0 \in R^n$ is called an **asymptotic value** of f if there exists a piecewise smooth curve $\sigma(t)$, $0 \leq t < \infty$, which extends to ∞ so that

$$\lim_{t\to\infty} f(\sigma(t)) = x_0;$$

$\sigma(t)$ is called an **asymptotic curve** of f.

If we use the coordinate notations $f = (f_1, \ldots, f_n)$, $x_0 = (x_{01}, \ldots, x_{0n})$ then we call x_{0j} an asymptotic value of the function f_j, $j = 1, \ldots, n$, and $\sigma(t)$ is called an **asymptotic curve** of f_j.

Remark 1: Asymptotic values correspond to infinitely far intersection points of the projective closures of the hypersurfaces $f_i(x_1, \ldots, x_n) = x_{0i}, i = 1, \ldots, n$.

PROPOSITION 1: *Let f: $R^n \to R^n$ be a C^1 map. If f has no asymptotic values then f satisfies condition* (2).

Proof: We assume in order to get a contradiction that f does not satisfy condition (2). Let $\sigma(t)$, $0 \le t < \infty$, be a curve such that $\lim_{t\to\infty} \| \sigma(t) \|_2 = \infty$ but the length of $f(\sigma(t))$, $0 \le t < \infty$, is $L < \infty$.

We verify the Cauchy condition for $f(\sigma(t))$:

$$\forall \epsilon > 0 \ \exists t(\epsilon) \text{ such that } \forall t_1, t_2 > t(\epsilon), \quad \| f(\sigma(t_1)) - f(\sigma(t_2)) \|_2 < \epsilon.$$

If not, then there is an $\epsilon_0 > 0$ and a sequence $0 < t_1 < t_2 < \cdots < t_m \to \infty$ such that

$$\| f(\sigma(t_{2j})) - f(\sigma(t_{2j-1})) \|_2 \ge \epsilon_0, \quad j = 1, 2, 3, \ldots.$$

But this implies the contradiction

$$L \ge \sum_{j=1}^{\infty} \| f(\sigma(t_{2j})) - f(\sigma(t_{2j-1})) \|_2 = \infty.$$

Hence $\lim_{t\to\infty} f(\sigma(t)) = x_0$ exists. But this contradicts the assumption on f.
∎

3. On the asymptotic curves of polynomials and polynomial maps over R

The real Jacobian conjecture [14] was the following assertion: *Let F: $R^n \to R^n$ be a polynomial map such that* $\det J(F)$ *never vanishes; then F is a global diffeomorphism of R^n onto R^n.*

This conjecture was shown to be false by S. Pinchuk [13]. Pinchuk's beautiful counterexample is of a polynomial map (P, Q) which, in fact, consists of two polynomials in the ring $R[t, h, f]$ where

$$t = XY - 1, \quad h = t(Xt + 1), \quad f = (Xt + 1)^2(h + 1)/X.$$

It is arranged so that $\det J(P,Q)$ is a sum of three squares of polynomials that have no common real zero. Hence $\det J(P,Q)$ is always positive.

If we define the following polynomials

$$u_1 = Y, \quad u_2 = XY, \quad u_3 = X^2Y - X,$$

then we can write the primitive polynomials that Pinchuk used as follows:

$$t = u_2 - 1, \quad h = (u_2 - 1)(u_3 + 1), \quad f = (u_3 + 1)^2((u_2 - 1)^2 + u_1).$$

Thus, the coordinate polynomials P, Q in the counterexample belong to the ring $R[u_1, u_2, u_3]$.

This last ring of polynomials has a very special property. In order to explain this property let us define the following birational map:

$$R(X,Y) = (X^{-1}, X + X^2Y).$$

Let us denote by $I(R)$ the set of all those polynomials $P(X,Y) \in R[X,Y]$ such that

$$(P \circ R)(X,Y) = P(X^{-1}, X + X^2Y) = A(X,Y) \in R[X,Y].$$

We say that $P(X,Y)$ satisfies an asymptotic identity with respect to the rational map R. Thus $I(R)$ consists of all those polynomials in $R[X,Y]$ that satisfy an asymptotic identity with respect to $R(X,Y)$.

Clearly, $u_1, u_2, u_3 \in I(R)$ and so $R[u_1, u_2, u_3] \subseteq I(R)$ (in fact we will see later that $R[u_1, u_2, u_3] = I(R)$). Hence also the coordinate polynomials in Pinchuk's construction are in $I(R)$.

If we take any pair of polynomials (P,Q) from $I(R)$, then they satisfy the following double asymptotic identity:

$$P(X^{-1}, X + X^2Y) = A(X,Y) \in R[X,Y],$$
$$Q(X^{-1}, X + X^2Y) = B(X,Y) \in R[X,Y].$$

Hence we have

$$\lim_{X\to 0}(P,Q)(X^{-1}, X + X^2Y) = \lim_{X\to 0}(A(X,Y), B(X,Y)) = (A(0,Y), B(0,Y)).$$

Thus all the points on the real algebraic (affine) curve $(A(0,Y), B(0,Y))$, $Y \in R$, are, in fact, asymptotic values of the map (P,Q) (they are realized by the rational curve $(X^{-1}, X + X^2Y)$ where $X \to 0$, $X \neq 0$ and Y is fixed).

So by Hadamard's Theorem the map (P,Q) cannot be a global diffeomorphism. As a consequence, in order to construct a counterexample to the real Jacobian conjecture, it suffices to find a Jacobian pair (P, Q) whose coordinate polynomials are in $I(R)$.

Hence all that one needs is a method to find $P, Q \in I(R)$ with $\det J(P, Q) > 0$ always. It is assured in advance that any such map is not injective. This was demonstrated nicely in Pinchuk's paper [13].

In this section we will show that in order to prove that a given polynomial map is injective, it suffices to exclude the existence of a certain type of asymptotic values. These asymptotic values are called either X-finite or Y-finite.

Then we will discuss the structure of the corresponding asymptotic curves. The map that was constructed by Pinchuk has the simplest Y-finite asymptotic curve.

Definition 2: An asymptotic curve $\sigma(t) = (X(t), Y(t))$, $0 \leq t < \infty$, of a polynomial $P(X,Y) \in R[X,Y]$ or of a polynomial map (P,Q) is called **X-finite** (**Y-finite**) if $\lim_{t\to\infty} | Y(t) |= \infty$ while $-\infty < \lim_{t\to\infty} X(t) < \infty$ ($\lim_{t\to\infty} | X(t) |= \infty$ while $-\infty < \lim_{t\to\infty} Y(t) < \infty$). The corresponding asymptotic value is called **an X-finite (Y-finite) asymptotic value**.

Remark 2: X- and Y-finite asymptotic values mean that the projective closure of the curve $P(X,Y)$ =const. (or the curves $P(X,Y)$ =const., $Q(X,Y)$ =const.) passes through the infinitely far point of intersection of the X-axis or the Y-axis.

Definition 3: A continuous real-valued function g: $[0,\infty) \to R$ is said to be **almost monotonic** if for any $r_0 \in R$ there is a $T(r_0)$ such that $g(t) \neq r_0$ for $t > T(r_0)$.

Remark 3: If g is almost monotonic then $\lim_{t\to\infty} g(t)$ exists. It might be ∞ or $-\infty$.

PROPOSITION 2: *Let f: $R^2 \to R^2$ be a polynomial map that satisfies the Jacobian condition*

$$\forall (X,Y) \in R^2, \quad \det J(f)(X,Y) \neq 0.$$

Let $\sigma(t) = (X(t), Y(t))$, $0 \leq t < \infty$, be an asymptotic curve of f.

Then $X(t)$ and $Y(t)$ are almost monotonic.

Proof: Let us prove that $Y(t)$ is almost monotonic. We assume, in order to get a contradiction, that there is a $r_0 \in R$ and a sequence $0 < t_1 < t_2 < \cdots < t_j \to \infty$ such that

$$Y(t_j) = r_0, \quad j = 1, 2, 3, \ldots.$$

Since $\lim_{t\to\infty}, \| \sigma(t) \|_2 = \infty$ it follows that

$$\lim_{j\to\infty} | X(t_j) | = \infty. \tag{3}$$

Let us represent the polynomial coordinates of $f(X,Y)$ as polynomials in X:

$$P(X,Y) = P_n(Y)X^n + \cdots + P_1(Y)X + P_0(Y),$$
$$Q(X,Y) = Q_m(Y)X^m + \cdots + Q_1(Y)X + Q_0(Y).$$

Since the limit

$$\lim_{j\to\infty} P(X(t_j), Y(t_j)) = \lim_{j\to\infty} P(X(t_j), r_0)$$

exists and is finite, it follows by (3) that

$$P_n(r_0) = \cdots = P_1(r_0) = 0.$$

Similarly, we obtain for $Q(X,Y)$

$$Q_m(r_0) = \cdots = Q_1(r_0) = 0.$$

But this implies that $\det J(f)(X, r_0) = 0$, which contradicts the assumption on $f(X,Y)$. ∎

PROPOSITION 3: *Let f: $R^2 \to R^2$, $f(X,Y) = (P(X,Y), Q(X,Y))$ be a polynomial map that satisfies the Jacobian condition*

$$\forall (X,Y) \in R^2, \quad \det J(f)(X,Y) \neq 0,$$

and that also satisfies

$$\deg P = \deg_X P + \deg_Y P \quad (\text{or } \deg Q = \deg_X Q + \deg_Y Q). \tag{4}$$

Then any asymptotic value of f is either X-finite or Y-finite.

Proof: Let $\sigma(t) = (X(t), Y(t))$, $0 \le t < \infty$ be an asymptotic curve of f. Then by Proposition 2 the limits $\lim_{t\to\infty} X(t)$, $\lim_{t\to\infty} Y(t)$ exist. By (4), however, one of these limits must be finite. ∎

Definition 4: Let f: $R^2 \to R^2$, $f(X,Y) = (P(X,Y), Q(X,Y))$ be a polynomial map. Let k be a non-negative integer such that

$$\deg_X P < 2k + 1. \tag{5}$$

The standard k-transformation (change of variables) is defined by

$$T_k\colon (X,Y) \to (U,(1+U^{2k})V). \tag{6}$$

The k-normalized form of f is the polynomial map

$$F_k = f \circ T_k\colon R^2 \to R^2,$$
$$F_k(U,V) = f(U,(1+U^{2k})V).$$

We now are ready to prove our reduction theorem.

THEOREM 1: *Let $f\colon R^2 \to R^2$ be a polynomial map that satisfies the Jacobian condition*

$$\forall (X,Y) \in R^2, \quad \det J(f)(X,Y) \neq 0.$$

Then also the k-normalized form of f, F_k, satisfies the Jacobian condition and f is a global diffeomorphism iff F_k has no X-finite or Y-finite asymptotic values.

Proof: Since $\det J(F_k)(U,V) = (1+U^{2k})\det J(f)(X,Y)$, it follows that F_k satisfies the Jacobian condition. We note that $T_k\colon R^2 \to R^2$ is a global diffeomorphism and so, since $F_k = f \circ T_k$, it follows that f is a global diffeomorphism iff F_k is a global diffeomorphism.

Thus f is a global diffeomorphism iff F_k has no asymptotic values.

We claim that F_k satisfies condition (4) of Proposition 3. If this will be proved, then the conclusion will follow by Proposition 3. To prove that F_k satisfies condition (4) we calculate degrees

$$\deg_{(U,V)} X^i Y^j = i + (2k+1)j,$$
$$\deg_U X^i Y^j = i + 2kj,$$
$$\deg_V X^i Y^j = j.$$

Next, we represent $P(X,Y)$ as follows:

$$P(X,Y) = a_{nm}X^nY^m + \sum_{\text{either } j<m \text{ or } j=m \text{ and } i<n} a_{ij}X^iY^j,$$

where $a_{nm} \neq 0$. By (5) and (6) we obtain

$$\deg_U X^iY^j \leq \deg_U X^nY^m,$$
$$\deg_V X^iY^j \leq \deg_V X^nY^m,$$
$$\deg_{(U,V)} X^iY^j \leq \deg_{(U,V)} X^nY^m,$$

from which it follows that $\deg_{(U,V)} P = \deg_U P + \deg_V P$, so F_k satisfies condition (4). ∎

This reduction theorem was proved in an effort to prove the real Jacobian conjecture. However, now we know the conjecture to be false. Nevertheless, it still motivates the study of asymptotic values and asymptotic curves of polynomial maps, in particular those that are X-finite or Y-finite.

PROPOSITION 4: *Let $P(X,Y) = P_n(Y)X^n + \cdots + P_1(Y)X + P_0(Y) \in R[X,Y]$ and let $\sigma(t) = (X(t), Y(t))$, $0 \le t < \infty$, be a Y-finite asymptotic curve of $P(X,Y)$ that corresponds to the Y-finite asymptotic value C. We denote $\lim_{t\to\infty} Y(t) = Y_0$. Then*

(i) $Y(t) = Y_0 + \sum_{j=1}^{N} A_j/X(t)^{\alpha_j/\beta_j} + o(X(t)^{-\alpha_N/\beta_N})$ *as* $t \to \infty$, *where* $1 \le N \le n$, $\alpha_1, \ldots, \alpha_N, \beta_1, \ldots, \beta_N \in Z^+$ *and* $\alpha_1/\beta_1 < \cdots < \alpha_N/\beta_N$.

(ii) $P(X, Y_0 + \sum_{j=1}^{N} A_j/X^{\alpha_j/\beta_j}) = C + O(X^{-\alpha_1/\beta_1})$ *as* $|X| \to \infty$.

(iii) $P_n(Y_0) = 0$.

(iv) $A_1, \ldots, A_{N-1}$ *depend on* $P(X,Y)$ *only and not on* C. A_N *depends on* $P(X,Y)$ *and on* C *in the following way: there is a polynomial* $q(X)$ *which depends on* $P(X,Y)$ *only such that* $q(A_N^{\beta_N}) = C$.

(v) *The number of possible values for* $(A_1, \ldots, A_{N-1})$ *is a finite number which depends on* $P(X,Y)$ *only. (It is less than or equal to* $(\deg_Y P(X,Y))^n n!$.*)*

Remark 4: The construction that we shall introduce below resembles the construction in Walker [18], page 98. See also [1]. The difference is due to the fact that R is not algebraically closed, and so we shall take advantage of the fact that R is ordered. Moreover, the correct bound in (v) above is $\deg_Y P(X,Y)$ [18], Theorem 3.2, page 106.

Proof: We shall use the short notation $X = X(t)$, $Y = Y(t)$ in the proof. By the following facts: $\lim X = \infty$, $\lim Y = Y_0$ is finite and $\lim P(X,Y) = C$ is finite, it follows that we have the following equations:

$$\begin{gathered} P_n(Y)X^n + \cdots + P_1(Y)X + P_0(Y_0) \to C, \\ P_n(Y)X^{n-1} + \cdots + P_1(Y_0) \to 0, \\ \vdots \\ P_n(Y)X + P_{n-1}(Y_0) \to 0, \\ P_n(Y_0) = 0. \end{gathered} \tag{7}$$

If $P_n(Y_0) = \cdots = P_1(Y_0) = 0$, then on writing $Y = Y_0 + \epsilon$ in the first equation we obtain

$$\epsilon(P_n'(Y_0)X^n + P_{n-1}'(Y_0)X^{n-1} + \cdots + P_1'(Y_0)X) + P_0(Y_0) + O(\epsilon^2) \to C$$

(if $P_n'(Y_0) = \cdots = P_1'(Y_0) = 0$ we use higher-order Taylor expansions of $P_j(Y_0+\epsilon)$ about Y_0). If $P_0(Y_0) = C$, then we easily see that $Y(t) \equiv Y_0$ satisfies (ii). Otherwise, $\epsilon = (C - P_0(Y_0))/(P_n'(Y_0)X^n + \cdots + P_1'(Y_0)X)$ will lead to the conclusions.

Thus we assume that not all of $P_n(Y_0), \ldots, P_1(Y_0)$ are zeros. We shall use the equations (7) from bottom to top in order to expand Y into a fractional power polynomial in X and estimate the error term. We denote by k_j, $1 \le j \le n$, the smallest integer for which $P_{n-j+1}^{(k_j)}(Y_0) \neq 0$. Clearly, $1 \le k_j \le \deg P_{n-j+1}$. If $P_{n-1}(Y_0) \neq 0$ then, by (7), $\lim (Y - Y_0)^{k_1} X = C_{11}$ is finite and different from 0. Thus we obtain

$$Y = Y_0 + A_1/X^{1/k_1} + o(X^{-1/k_1}) \quad \text{as } |X| \to \infty.$$

If $P_{n-1}(Y_0) = 0$, we conclude that $C_{11} = 0$ and proceed to the previous equation in (7).

If $P_{n-2}(Y_0) \neq 0$, we denote

$$\lim(Y - Y_0)^{k_1} X^2 = C_{21}, \quad \lim(Y - Y_0)^{k_2} X = C_{22}.$$

If both C_{21} and C_{22} are finite, then by (7) they satisfy the following:

$$(P_n^{(k_1)}(Y_0)/k_1!)C_{21} + (P_{n-1}^{(k_2)}(Y_0)/k_2!)C_{22} + P_{n-2}(Y_0) = 0,$$

and the following cases are possible:

$$C_{21} \neq 0 \text{ and } C_{22} = 0, \text{ then } k_1 < 2k_2;$$
$$C_{21} = 0 \text{ and } C_{22} \neq 0, \text{ then } k_1 > 2k_2;$$
$$C_{21} \neq 0 \text{ and } C_{22} \neq 0, \text{ then } k_1 = 2k_2 \text{ and } C_{21} = C_{22}^2.$$

The case $C_{21} = C_{22} = 0$ is not possible, and so is the case where one is finite while the other is not. If both C_{21} and C_{22} are not finite, then it must be that $(Y - Y_0)^{k_1} X \approx (Y - Y_0)^{k_2}$ so that $k_2 < k_1$ and the limit $\lim(Y - Y_0)^{k_1-k_2} X$ is finite non-zero.

If $P_{n-2}(Y_0) = 0$, we proceed to the previous equation in (7).

However, this algorithm must terminate after at most $n-1$ steps for otherwise $P_n(Y_0) = \cdots = P_1(Y_0) = 0$. Thus we conclude that there are two integers α_1, β_1 such that $\lim(Y - Y_0)^{\alpha_1} X^{\beta_1} = C_1 \neq 0$ and is finite. C_1 is obtained by solving for $(Y - Y_0)$ an equation of the form

$$(P_n^{(k_1)}(Y_0)/k_1!)(Y - Y_0)^{k_1} X^s + \cdots + (P_{n-s+1}^{(k_s)}(Y_0)/k_s!)(Y - Y_0)^{k_s} X + P_{n-s}(Y_0) = 0 \text{ or } C,$$

depending on whether $s = N$ or not, where $P_{n-s}(Y_0) \neq 0$ and where only those terms $(Y - Y_0)^{k_j} X^i$ for which i/k_j is maximal count. Hence

$$Y = Y_0 + A_1 X^{-\alpha_1/\beta_1} + o(X^{-\alpha_1/\beta_1}) \quad \text{as } |X| \to \infty.$$

If $s < n$, we re-do the process with $Y - Y_0 - A_1 X^{-\alpha_1/\beta_1}$ instead of $Y - Y_0$. We note that none of the last $s + 1$ equations in (7) can determine A_2 and so the process starts effectively with

$$P_n(Y) X^{s+1} + \cdots + P_{n-s-1}(Y_0) \to 0 \text{ or } C,$$

depending on whether $s = N - 1$ or not.

This time, the output of the process gives information on linear combinations of elements of the form $(Y - Y_0)^b (X^{-\alpha_1/\beta_1})^{\beta_1 - b} (X^{\alpha_1 + a})$. Thus we get

$$Y = Y_0 + A_1 X^{-\alpha_1/\beta_1} + A_2 X^{-\alpha_2/\beta_2} + o(X^{-\alpha_2/\beta_2}) \quad \text{as } |X| \to \infty,$$

where $\alpha_2/\beta_2 > \alpha_1/\beta_1$.

We proceed till all the equations in (7) have been used. Finally, we note that the last coefficient A_N is either obtained with the aid of the top equation of (7) ($s = N$) or that this equation is a consequence of the previous ones. Since each $A_k^{\beta_k}$ is a solution of a different polynomial equation of degree d_k in X while $\beta_k \leq \deg_Y P(X, Y)$, and since obviously

$$\max\{d_1 \ldots d_N \mid 1 \leq d_k \leq n \text{ are different}\} \leq n!/(n - N)! \leq n!,$$

the number of possible different values for $(A_1, \ldots, A_{N-1})$ is at most $(\deg_Y P(X, Y))^n n!$. ∎

Remark 5: The error term $O(X^{-\alpha_1/\beta_1})$ in (ii) is a finite linear combination of elements of the form $X^{-\alpha/\beta}$.

Remark 6: The last proposition can also be proved using Puiseux expansions for local branches of the projective closure of the curve $P(X, Y) = C$, centered at the infinitely far point of the X-axis.

We are now ready to prove our theorem on the equivalence between the fact that $P(X, Y)$ has a Y-finite asymptotic value and the fact that $P(X, Y)$ should satisfy a corresponding asymptotic identity.

THEOREM 2: *Let $P(X, Y) = P_n(Y) X^n + \cdots + P_0(Y) \in R[X, Y]$. A number $C \in R$ is a Y-finite asymptotic value of $P(X, Y)$ iff there are integers $k \geq 1$,*

$1 \le N \le n, 1 \le a_1 < \cdots < a_N$ *real numbers,* $A_0, \ldots, A_{N-1} \in R$ *and polynomials* $B_0(Y), \ldots, B_M(Y) \in R[Y]$ *such that the following identity holds true:*

$$\begin{aligned} P(X^{-k}, A_0 + A_1 X^{a_1} + \cdots + A_{N-1} X^{a_{N-1}} + Y X^{a_N}) \\ = B_0(Y) + B_1(Y) X + \cdots + B_M(Y) X^M \end{aligned} \tag{8}$$

and $C = B_0(Y_0)$ *for some* $Y_0 \in R$.

Proof: One direction follows from Proposition 4 and from the remark that follows its proof (after substituting X^{-k} for X in (ii) of that proposition with a large enough k).

The other direction is easier, for if (8) holds true and if we take

$$\sigma(X) = (X^{-k}, A_0 + \cdots + A_{N-1} X^{a_{N-1}} + Y X^{a_N}),$$

for X near 0, then $\sigma(X)$ is a Y-finite asymptotic curve of $P(X, Y)$ and $\lim_{X \to 0} P(\sigma(X)) = B_0(Y)$, which shows that $B_0(Y)$ is the corresponding Y-finite asymptotic value of $P(X, Y)$. ∎

Remark 7: The smallest value of k in the last theorem depends on the Y-finite asymptotic curve only. It has the purpose of getting rid of the denominators $\beta_1, \ldots, \beta_N$ in (ii) of Proposition 4.

Thus, in particular, if $P, Q \in R[X, Y]$ share such an asymptotic curve then they will have the same smallest possible exponent k.

COROLLARY 1: *Let* $P(X, Y) \in R[X, Y]$. *Then* $(X(t), Y(t))$, $0 \le t < \infty$, *is a* Y*-finite asymptotic curve of* $P(X, Y)$ *that corresponds to the asymptotic value* C *iff* $(-X(t), Y(t))$ *is a* Y*-finite asymptotic curve of* $P(X, Y)$ *that corresponds to the same* C.

Definition 5: Let $P(X, Y) \in R[X, Y]$ have Y-finite asymptotic values. The identity (8) with the smallest possible value of k will be called a Y-**finite asymptotic identity** of $P(X, Y)$. The polynomial $A(X, Y) = B_0(Y) + \cdots + B_M(Y) X^M$ will be called the **dual polynomial** of $P(X, Y)$ (with respect to the identity (8)). The polynomial $A_0 + \cdots + A_{N-1} X^{N-1} + Y X^N$ will be called the **asymptotic coordinate** of $P(X, Y)$ (with respect to the identity (8)). k will be called the **exponent** of the Y-finite asymptotic curve of $P(X, Y)$.

Remark 8: We recall from the proof of Proposition 4 that A_0 must be a zero of $P_n(Y)$ and that there are at most $(\deg_Y P(X, Y))^n n!$ different asymptotic coordinates of $P(X, Y)$.

Finally, the Y-finite asymptotic values of $P(X,Y)$ coresponding to the Y-finite asymptotic curve that induces (8) are precisely $B_0(Y)$, $Y \in R$.

THEOREM 3 (The Duality Theorem): *If $P(X,Y) \in R[X,Y]$ has a Y-finite asymptotic curve of exponent k with the asymptotic coordinate*

$$A_0 + A_1 X^{a_1} + \cdots + A_{N-1} X^{a_{N-1}} + Y X^{a_N},$$

and with the dual polynomial $A(X,Y)$, then $A(X,Y)$ has a Y-finite asymptotic curve of exponent 1 with the asymptotic coordinate

$$-A_{N-1} X^{a_N - a_{N-1}} - \cdots - A_1 X^{a_N - a_1} + Y X^{a_N},$$

and with the dual polynomial $P(X^k, Y + A_0)$.

Proof: The inverse map of

$$U = X^{-k}, \quad V = A_0 + \cdots + A_{N-1} X^{a_{N-1}} + Y X^{a_N},$$

is given by

$$X = U^{-1/k}, \quad Y = -A_{N-1} U^{(a_N - a_{N-1})/k} - \cdots - A_1 X^{(a_N - a_1)/k} + (V - A_0) U^{a_N/k}.$$

Since $P(X^{-k}, A_0 + \cdots + A_{N-1} X^{a_{N-1}} + Y X^{a_N}) = A(X,Y)$, we obtain after replacing U by U^k and $V - A_0$ by V

$$P(U^k, V + A_0) = A(U^{-1}, -A_{N-1} U^{a_N - a_{N-1}} - \cdots - A_1 U^{a_N - a_1} + V U^{a_N}),$$

which is a V-finite asymptotic identity of $A(U,V)$ with the parameters that are described in the statement of the theorem. ∎

COROLLARY 2: *If $A(X,Y) = B_0(Y) + \cdots + B_M(Y) X^M$ is a dual polynomial of some $P(X,Y)$, then $B_M(0) = 0$.*

Proof: By the dual asymptotic identity given in the proof of the Duality Theorem we see that the free term of the asymptotic coordinate of $A(X,Y)$ is 0. This, however, must be a zero of $B_M(Y)$ by the remark that precedes the statement of the Duality Theorem. ∎

COROLLARY 3: *If $P(X,Y) \in R[X,Y]$ satisfies the Y-finite asympotic identity (8), then*

$$\lim_{U \to \infty} U^{-a_N/k} P_V(U,V) = B_0'(Y),$$

$$\lim_{U \to \infty} U^{-(a_N - a_1 - k)/k} P_U(U,V) = a_1 A_1 B_0'(Y)/k,$$

where as usual

$$U = X^{-k}, \quad V = A_0 + A_1 X^{a_1} + \cdots + A_{N-1} X^{a_{N-1}} + Y X^{a_N}.$$

Proof: Differentiation of (8) with respect to Y gives the first limit on letting $|X| \to 0$.

Differentiation of (8) with respect to X yields

$$-kX^{a_{N-1}-a_1-k} P_U(U,V) + (a_1 A_1 + a_2 A_2 X^{a_2-a_1} + \cdots + a_N Y X^{a_N-a_1}) X^{a_N} P_V(U,V)$$
$$= X^{a_N-a_1+1}(B_1(Y) + \cdots + M B_M(Y) X^{M-1}).$$

On letting $|X| \to 0$ we get, using the first limit,

$$-k \lim_{|X|\to 0} X^{a_N-a_1-k} P_U(U,V) + a_1 A_1 B_0'(Y) = 0,$$

which proves the second equation. ∎

We now turn to polynomial maps over R. Suppose that

$$f(X,Y) = (P(X,Y), Q(X,Y))$$

is a polynomial map that has a Y-finite asymptotic value. According to what we have established the corresponding curve induces an asymptotic identity for $P(X,Y)$ and an asymptotic identity for $Q(X,Y)$ that share the same exponent and the same asymptotic coordinates (for these come from the fractional power expansion of the curve, and this expansion is unique). Thus we have

$$\begin{aligned} &P(X^{-k}, A_0 + A_1 X^{a_1} + \cdots + A_{N-1} X^{a_{N-1}} + Y X^{a_N}) = A(X,Y) \in R[X,Y], \\ &Q(X^{-k}, A_0 + A_1 X^{a_1} + \cdots + A_{N-1} X^{a_{N-1}} + Y X^{a_N}) = B(X,Y) \in R[X,Y]. \end{aligned} \tag{9}$$

We shall say that the map $f(X,Y)$ satisfies **the double asymptotic identity** (9).

Remark 9: The map that was constructed by S. Pinchuk [13] satisfies a simple double asymptotic identity. Its parameters are $k=1$, $A_0 = 0$, $A_1 = 1$, $a_1 = 1$, $a_2 = 2$, $N = 2$.

We are now ready to state a theorem on the geometric structure of the loci of the asymptotic values of a polynomial map over R, and to prove it.

THEOREM 4: *Let $f(X,Y)$ be a polynomial map over R. Then the set of all the asymptotic values of $f(X,Y)$ consists of a finite number of real plane (affine) algebraic curves. These curves have one of the following three forms:*

(a) $\{(A(X), B(X)) \mid X \in R\}$ *where* $A, B \in R[X]$.
(b) $\{(A(X), B(X)/(1+A(X)^{2k})) \mid X \in R\}$ *where* $A, B \in R[X]$, $k \in Z^+$.
(c) $\{(A(X)/(1+B(X)^{2k}), B(X)) \mid X \in R\}$ *where* $A, B \in R[X]$, $k \in Z^+$.

The number of the curves as well as the degrees $\deg A(X)$, $\deg B(X)$ *can be bounded by bounds that depend on* $\deg f$ *only.*

Proof: Let us consider the k-normalized form of f, F_k. As we proved already in Theorem 1, F_k can have only X-finite or Y-finite asymptotic curves. We shall show that the set of all the asymptotic values of F_k consists of finitely many algebraic curves of the form (a). Since $F_k = f \circ T_k$ and T_k is invertible with an inverse map $(X, Y/(1+X^{2k}))$ (or $(X/(1+Y^{2k}), Y)$ if we use for T_k the map $(U(1+V^{2k}), V)$), this will imply that the set of all the asymptotic values of f consists of finitely many curves of types (a), (b) or (c) as desired. Thus we may assume that our map has only Y-finite asymptotic values. We recall that if a Y-finite asymptotic curve is such that $Y_0 = \lim_{t\to\infty} Y(t)$, then Y_0 is a common zero of $P_n(Y)$ and $Q_m(Y)$ where

$$f(X,Y) = (P_n(Y)X^n + \cdots + P_0(Y), Q_m(Y)X^m + \cdots + Q_0(Y)).$$

So there are at most $\min\{\deg P_n, \deg Q_m\}$ possible values for Y_0. By Theorem 2 any Y-finite asymptotic value (a, b) is of the form $(A(0, Y_1), B(0, Y_1))$ for some $Y_1 \in R$, where $A(X,Y)$, $B(X,Y)$ are dual polynomials to the components $P(X,Y)$, $Q(X,Y)$ of the map $f(X,Y)$, and conversely, any point $(A(0,Y), B(0,Y))$ is a Y-finite asymptotic value of $f(X,Y)$.

As we already know,

$$\deg A(0,Y) \le \deg_Y P(X,Y), \quad \deg B(0,Y) \le \deg_Y Q(X,Y).$$

Also, the number of different $A(X,Y)$ is at most $(\deg_Y P(X,Y))^n n!$ and the number of different $B(X,Y)$ is at most $(\deg_Y Q(X,Y))^m m!$. These complete the proof of the theorem. ∎

We end this section by pointing out two properties of Y-finite asymptotic identities which are satisfied by maps having a constant non-zero Jacobian.

PROPOSITION: *Let* $f(X,Y) = (P(X,Y), Q(X,Y))$ *be a polynomial map such that* $\det J(f)(X,Y) \equiv 1$ *and such that* $f(X,Y)$ *satisfies the double asymptotic identity* (9). *Then:*

(a) *The polynomials* $\partial A/\partial Z(Z,W)$, $\partial B/\partial Z(Z,W)$ *considered over* C *have no common zero of the form* (Z_0, W_0), *where* $Z_0 \neq 0$.

(b) $k+1 \leq a_N$, *and there is equality iff* $(A(X,Y), B(X,Y))$ *is a Jacobian pair.*

Proof: To see that (a) holds true we differentiate (9) with respect to X and replace (X,Y) by the complex (Z,W) in accordance with the permanence principle. We get

$$-kZ^{-(k+1)}P_Z + (a_1A_1Z^{a_1-1} + \cdots + a_NWZ^{a_N-1})P_W = \partial A/\partial Z(Z,W),$$
$$-kZ^{-(k+1)}Q_Z + (a_1A_1Z^{a_1-1} + \cdots + a_NWZ^{a_N-1})Q_W = \partial B/\partial Z(Z,W).$$

Plugging in $(Z,W) = (Z_0, W_0)$ and assuming that

$$\partial A/\partial Z(Z_0,W_0) = \partial B/\partial Z(Z_0,W_0) = 0, \quad Z_0 \neq 0$$

will contradict the assumption $\det J(f)(Z,W) \equiv 1$.

To see that (b) holds true we note that, by the chain rule, we have

$$\det J(A,B)(X,Y) = -kX^{a_N-k-1} \det J(f)(U,V),$$

so by the assumption on $f(X,Y)$,

$$\det J(A,B) = -kX^{a_N-k-1},$$

and (b) follows. ∎

Along the lines of property (b) we can add a few more inequalities that are related to asymptotic identities.

PROPOSITION:

(c) *If* $P(X,Y) \in R[X,Y]$ *(or* $C[X,Y]$*) and satisfies the asymptotic identity* (8), *then*

$$M \leq a_N \deg_Y P(X,Y), \quad k \deg_X P(X,Y) \leq a_N \deg_Y P(X,Y).$$

(d) *If* $f(X,Y) = (P(X,Y), Q(X,Y))$ *is a polynomial map that satisfies the condition* $\det J(f)(X,Y) \neq 0\ \forall (X,Y) \in R^2$ *(or* C^2*) and satisfies the double asymptotic identity* (9), *then*

$$a_N \leq \max\{\deg_X A(X,Y), \deg_X B(X,Y)\}.$$

Proof: To see that (c) is true we write

$$P(X^k, A_0 + A_1/X^{a_1} + \cdots + A_{N-1}/X^{a_{N-1}} + Y/X^{a_N}) = B_0(Y) + \cdots + B_M(Y)/X^M.$$

If $P(U,V)=\sum a_{ij}U^iV^j$, $1\le i\le \deg_U P(U,V)$, $1\le j\le \deg_V P(U,V)$, then

$$\sum a_{ij}X^{ki}(A_0+A_1/X^{a_1}+\cdots+Y/X^{a_N})^j=B_0(Y)+\cdots+B_M(Y)/X^M.$$

Hence $M=a_Nj_0-ki_0\le a_Nj_0\le a_N\deg_Y P(X,Y)$, which proves the first inequality of (c). If

$$V=A_0+A_1/X^{a_1}+\cdots+A_{N-1}/X^{a_{N-1}}+Y/X^{a_N},$$

then $Y=X^{a_N}V+\phi(X)$, where $\phi(X)=-X^{a_N}(A_0+\cdots+A_{N-1}/X^{a_{N-1}})$. So

$$P(X^k,V)=$$
$$B_0(X^{a_N}V+\phi(X))+B_1(X^{a_N}V+\phi(X))/X+\cdots+B_M(X^{a_N}V+\phi(X))/X^M.$$

Hence $k\deg_X P(X,Y)=a_N\deg B_{j_0}-j_0\le a_N\deg B_{j_0}\le a_N\deg_Y P(X,Y)$.

To see that (d) is true we will denote

$$A(X,Y)=B_0(Y)+B_1(Y)X+\cdots+B_M(Y)X^M,$$
$$B(X,Y)=C_0(Y)+C_1(Y)X+\cdots+C_L(Y)X^L.$$

Then by (9) with the same $\phi(X)$ as above we have

$$P(X^k,Y)=\sum_{j=0}^{M}X^{-j}B_j(X^{a_N}Y+\phi(X)),$$
$$Q(X^k,Y)=\sum_{j=0}^{L}X^{-j}C_j(X^{a_N}Y+\phi(X)).$$

Hence

$$\partial P/\partial Y(X^k,Y)=\sum_{j=0}^{M}X^{a_N-j}B_j'(X^{a_N}Y+\phi(X)),$$
$$\partial Q/\partial Y(X^k,Y)=\sum_{j=0}^{L}X^{a_N-j}C_j'(X^{a_N}Y+\phi(X)).$$

So if $a_N>\max\{M,L\}$ we would get $\partial P(0,y)/\partial Y(0,Y)=\partial Q/\partial Y(0,Y)=0$. ∎

4. Asymptotic identities in the complex case

For this case of an algebraically closed field we will use the method developed in [18]. See also [1]. A key theorem concerning parametrizations of algebraic curves can be formulated as follows

THEOREM 3.1 ([18], page 98): *$K(x)^*$ is algebraically closed.*

The proof of that theorem given in [18] has the advantage of being constructive, and we shall use this method to construct asymptotic identities for complex polynomials. We shall refer to this construction as Newton's algorithm.

An important device in Newton's algorithm is a certain Newton polygon which differs from the standard Newton polygon; see [1], [7]. In the next section we shall explore the relations between the two types of polygons and apply asymptotic identities in order to prove a result of J. Lang [7].

The typical identities we shall get are described as follows.

Definition 6: Let $P(X,Y) \in C[X,Y]$, $b \in C$. **An X-asymptotic identity of $P(X,Y)$ at level** b is an identity of the following form:

$$P((X-b)^k+b,(X-b)^{-l}(A_0+A_1(X-b)+\cdots+A_{N-1}(X-b)^{N-1}+Y(X-b)^N)) = A(X,Y),$$

where $k,l \in Z^+$, $A_0,\ldots,A_{N-1} \in C$, $A(X,Y) \in C[X,Y]$. $A(X,Y)$ is called the **dual polynomial** of $P(X,Y)$ with respect to the identity.

Remark 10: If $P(X,Y)$ satisfies an X-asymptotic identity at level b with a dual polynomial $A(X,Y)$, then this identity induces the following set of asymptotic values of $P(X,Y)$:

$$\{A(b,Y) \mid Y \in C\}.$$

Let $P(X,Y) = P_n(X)Y^n + \cdots + P_1(X)Y + P_0(X) \in C[X,Y]$ satisfy the following condition:

There exists an m, $0 < m < n$, such that

$$O(P_m(X)) < O(P_n(X)). \tag{10}$$

Here, as in [18], page 89, $O(P_m(X))$, the order of $P_m(X)$, is the smallest effective degree of X in $P_m(X)$.

Let $a \in C$. Then by Theorem 3.2 [18], page 106, there exists a unique set of elements $Y_1(X,a),\ldots,Y_n(X,a)$ in $C(X)^*$ such that

$$P(X,Y) = P_n(X)\prod_{j=1}^{n}(Y - Y_j(X,a)) + a.$$

Thus for each $j = 1, 2, \ldots, n$ we have a representation

$$Y_j(X, a) = X^{-u_j/v_j} \sum_{i=0}^{\infty} \alpha_{ij}(a) X^{i/v_j},$$

where $v_j \in Z^+$, $u_j \in Z$ and $\alpha_{ij}(a)$ are algebraic in a. The Newton polygon of Walker (whose vertices are $(j, O(P_j(X)))$, $j = 0, \ldots, n$) must have segments of positive slope (by condition (10)).

Remark 11: Given $a \in C$, we could replace condition (10) by

$$O(P_0(X) - a) < O(P_n(X)) \tag{11}$$

in order to get the same conclusion about the Newton polygon.

Hence there exists a j_0, $1 \le j_0 \le n$, such that $u_{j_0} \in Z^+$. From now on we will fix $j = j_0$. We have the following identity:

$$P(X, X^{-u_{j_0}/v_{j_0}} \sum_{i=0}^{\infty} \alpha_{ij_0}(a) X^{i/v_{j_0}}) \equiv a.$$

We can think of the last identity more precisely as follows. For each $s \in Z^+$ there exist $m(s), n(s) \in Z$ such that

$$P(X, X^{-u_{j_0}/v_{j_0}} \sum_{i=0}^{s} \alpha_{ij_0}(a) X^{i/v_{j_0}}) = a + \sum_{i=n(s)}^{m(s)} b_i(a) X^{i/L},$$

for some fixed $L \in Z^+$ (in fact $1 \le L \le v_{j_0}$) and fixed algebraic functions $b_i(a)$, where $\lim_{s\to\infty} n(s) = \infty$.

Let $s = N$ be the first for which $n(N) > 0$. We note that in Newton's algorithm [18], page 98, the number $P_0(0) - a = a_0$ was involved in computing the values of c from $\sum a_h c^h = 0$ for all the stages $s < N$. Hence $\alpha_{ij_0}(a)$ are numbers independent of a for $0 \le i \le N-1$. a_0 was involved in the computations for the first time when $s = N$, where we had an equation of the form

$$(P_0(0) - a) + \sum a_h c^h = 0$$

to solve. If we denote $Y = c$, we obtain

$$a = P_0(0) + \sum a_h Y^h = B_0(Y) \in C[Y].$$

Finally, we replace X by, say, $X^{v_{j_0} L}$. Thus we can formulate the following

THEOREM 5: *Let $P(X,Y) = P_n(X)Y^n + \cdots + P_1(X)Y + P_0(X) \in C[X,Y]$ satisfy $0 < O(P_n(X))$. Then $P(X,Y)$ satisfies X-asymptotic identities at level 0,*

$$P(X^k, X^{-l}(A_0 + A_1X + \cdots + A_{N-1}X^{N-1} + YX^N))$$
$$= B_0(Y) + B_1(Y)X + \cdots + B_M(Y)X^M \in C[X,Y],$$

The number of such identities with minimal k, N is at most $\deg_Y P(X,Y)$.

Remark 12: Conditions (10) and (11) are essential for the finiteness assertion of the theorem. For example, let $P(X,Y) = XY - X^2$. Then the only X-asymptotic identity satisfied by $P(X,Y)$ is $P(X, Y/X) = Y - X^2$. Any $a \neq 0$ is an asymptotic value of $P(X,Y)$ which is attained along one of the canonical curves, i.e., $\{(X, a/X) \mid X \in C^*\}$. However, $a = 0$ is not attained in that manner, for the curve $\{(X,0) \mid X \in C^*\}$ is bounded near $X = 0$.

But $a = 0$ is an X-finite asymptotic value at level 0. There is a wealth of non-canonical asymptotic curves that realize 0. In fact any curve $Y = Y(X)$ for which $\lim_{X\to 0} \mid Y(X) \mid = \infty$, $\lim_{X\to 0} XY(X) = 0$ will do.

THEOREM 6: *Let $P(X,Y) = P_n(X)Y^n + \cdots + P_1(X)Y + P_0(X) \in C[X,Y]$. Then for any zero b of $P_n(X)$, $P(X,Y)$ satisfies X-asymptotic identities at level b and their number (with minimal k, N) is at most $\deg_Y P(X,Y)$.*

Conversely, if $P(X,Y)$ satisfies a X-asymptotic identity at level b, then $P_n(b) = 0$. In particular, the total number of X-asymptotic identities with minimal k, N is at most $\deg P_n(X) \cdot \deg_Y P(X,Y) \leq \deg_X P(X,Y) \cdot \deg_Y P(X,Y)$.

Proof: The case where $O_{(X-b)}(P_m(X)) < O_{(X-b)}(P_n(X))$ for some $0 \leq m < n$ is taken care of as in the proof of the previous theorem. So let us assume that $O_{(X-b)}(P_n(X)) \leq O_{(X-b)}(P_m(X))$ for all $0 \leq m < n$. We shall denote $L = O_{(X-b)}(P_n(X))$. Then our assumption is equivalent to

$$P(X,Y) = C(X-b)^L Y^n + (X-b)^{L+1} Q(X,Y),$$

where $C = C(X) \in C[X]$, $C(b) \neq 0$, $Q(X,Y) \in C[X,Y]$. Let $a \in C^*$ and consider the n rational functions of $(X-b)^{1/n}$,

$$Y_k(X,a) = \epsilon_k (a/C)^{1/n} (X-b)^{-L/n}, \quad k = 0, 1, \ldots, n-1,$$

where ϵ_k are the n roots of unity of order n. Then

$$\lim_{X\to b} P(X, Y_k(X,a)) = a, \quad k = 0, 1, \ldots, n-1,$$

and we can induce n asymptotic identities at level b. Finally, if $P_n(b) \neq 0$, then since $(X-b)^{-l}(A_0 + A_1(X-b) + \cdots + A_{N-1}(X-b)^{N-1} + Y(X-b)^N)$ has a pole at $X = b$ it follows that

$$\lim_{X \to b} | P((X-b)^k + b, (X-b)^{-l}(A_0 + \cdots + Y(X-b)^N)) | = \infty,$$

so that $P(X,Y)$ cannot satisfy X-asymptotic identities at level b. ∎

Remark 13: It follows from the previous theorem that there are polynomials with no X or Y asymptotic values. Obvious examples are $X - Y$, $X - Y^2$. But still, one can mimick the procedure of Newton's algorithm for the expansions of the asymptotic curves.

For example, let us consider $P(X,Y) = Y^2 + X - X^2$. Consideration of the level curves $P(X, Y_1) = a$ leads to

$$Y_1(X,a) = \pm X(1 + (a-X)/X^2)^{1/2} = \pm X(1 + (a-X)/2X^2 + \cdots).$$

So using the same idea of truncation to get

$$Y(X,a) = \pm X(1 + (a - X - 1/4)/2X^2),$$

we obtain

$$P(X, Y(X,a)) = a - (a - 1/4)/2X + \cdots,$$

which is an asymptotic identity for $P(X,Y)$.

In fact, we have expanded above an algebraic function into Laurent series (in $1/X$ in this example).

This can be done in general, as will follow from the next lemma.

LEMMA 1: *Let $P(X,Y) = P_n(X)Y^n + \cdots + P_1(X)Y + P_0(X) \in C[X,Y]$, $P_n(X) \not\equiv 0$. Then there exists a unique set of elements $Y_1, \ldots, Y_n$ in $C(X^{-1})^*$ such that*

$$P(X,Y) = P_n(X) \prod_{j=1}^{n} (Y - Y_j).$$

Proof: Let us denote $T = X^{-1}$. Then $P_j \in C(T)^*$ and so, by Theorem 3.2 [18], page 106, there is a unique set of elements $Z_1, \ldots, Z_n \in C(T)^*$ such that $P(X,Y) = P_n(X) \prod_{j=1}^{n} (Y - Z_j)$. Substitution of X^{-1} for T in the Z_j gives the result. ∎

Remark 14: The above lemma cannot be extended to the general case where $P_j(X) \in C(X)^*$ (as in Walker's theorem). For example, if

$$P(X,Y) = Y - \sum_{n=0}^{\infty} X^n/n!$$

then there is no element $Y_1 \in C(X^{-1})^*$ such that $Y_1 - \sum_{n=0}^{\infty} X^n/n! = 0$.

A consequence of the lemma, which is proved in a similar manner to that of the previous two theorems, yields the following

THEOREM 7: *Let $P(X,Y) = P_n(X)Y^n + \cdots + P_1(X)Y + P_0(X) \in C[X,Y]$, $b \in C$ be such that $P_n(b) \neq 0$. Then $P(X,Y)$ satisfies asymptotic identities of the form*

$$P((X-b)^{-k}, (X-b)^{-l}(A_0 + A_1(X-b) + \cdots + A_{N-1}(X-b)^{N-1} + Y(X-b)^N)) \in C[X,Y].$$

The number of such identities with minimal k, N does not exceed $\deg_Y P(X,Y)$.

Remark 15: It was pointed out to me by Prof. B. Wajnryb that it is known that the complex case can also be reduced to the case where the polynomial map has only X-finite or Y-finite asymptotic curves and no others (just like in the real case that was treated on the previous section). We should mention that such a reduction cannot be acomplished via the k-standard change of variables T_k as in the real case (for T_k is not an automorphism of $C[X,Y]$). Also, we point out that the general asymptotic identity of the last theorem reduces to the Y-finite asymptotic identity (8) when $k \in Z^+$ and $l = 0$.

5. Application to Newton polygons of Jacobian pairs

We now recall some definitions and results related to Newton polygons and Jacobian pairs.

Let $P(X,Y) = \sum a_{ij} X^i Y^j \in C[X,Y]$. We will denote

$$S(P) = \{(i,j) \mid a_{ij} \neq 0\} \cup \{(0,0)\},$$

and $N(P)$ will denote the smallest convex subset of R^2 containing $S(P)$; $N(P)$ is called the **Newton polygon** of $P(X,Y)$.

Another notion of a Newton polygon played a role in the construction of Walker [18], page 98. There we considered $P(X,Y) = a_0 + a_1 Y + \cdots + a_n Y^n$, where $a_j \in C(X)^*$ and $a_n \not\equiv 0$. If $\alpha_j = O(a_j)$, then we plotted the points (j, α_j) and considered the convex polygonal arc each of whose vertices is a (j, α_j) and such that no (i, α_i) lies below the arc.

We shall call this arc **Walker's boundary** and denote it by $W(P)$. We note that in the case $P(X,Y) \in C[X,Y]$ the vertices of $W(P)$ are in $S(P)$, but in some sense these are the points of $S(P)$ closest to the origin while the points of $N(P) \cap S(P) - \{(0,0)\}$ are those farthest from the origin.

It follows from the definitions that

$$W(P) =$$

$$(\partial(N(P) - \text{Convexhull } (S(P) - \{(0,0)\})) - \{\text{edges of } N(P) \text{ containing } (0,0)\})$$

$$\cup \{\text{edges of } N(P) \text{ of a positive slope}\} \cup \{\text{edges of } W(P) \text{ of slope } 0, \infty\}.$$

There are various results on Newton polygons and Jacobian pairs. Let us recall some of them.

THEOREM A (Abhyankar, [1]): *Every Jacobian pair is an automorphic pair iff the Newton polygons of every Jacobian pair are triangles with vertices on the coordinate axes.*

THEOREM B (McKay and Wang, [8]): *The Newton polygons of an automorphic pair are similar triangles.*

THEOREM C (J. Lang, [7]): *The Newton polygons of a Jacobian pair are similar polygons.*

THEOREM D (J. Lang, [7]): *Every Jacobian pair is an automorphic pair iff the Jacobian condition implies that $N(P)$ has no edge of a positive slope.*

All of these results are results on $N(P)$. Walker's boundary $W(P)$ is not mentioned there explicitly. This situation is interesting, because by Hadamard's Theorem (Section 2) a Jacobian pair that is also an automorphic pair cannot have asymptotic values. Such a pair has asymptotic values iff its components satisfy a double asymptotic identity. However, such identities are partially determined by $W(P)$.

The explanation lies in the fact that the intersection of the sets of edges of slopes different from 0 or ∞ of $N(P)$ and of $W(P)$ is exactly the set of edges of positive slope, and these edges are those whose existence assures that the fractional power expansion of Walker corresponds to an asymptotic curve (and not to a compact curve) (see the previous section, conditions (10) and (11)).

Note that the condition of Theorem D (positive slope) is weaker than the condition of Theorem A (triangles with vertices on the coordinate axes). Let us give a proof for this stronger theorem, Theorem D, that is based on asymptotic identities.

THEOREM D (J. Lang, [7]): *Every Jacobian pair is an automorphic pair iff the Jacobian condition implies that $N(P)$ has no edge of a positive slope.*

Proof (based on asymptotic identities): One direction follows by Theorem B:

If every Jacobian pair is an automorphic pair and if (P,Q) is such a pair, then by Theorem B, $N(P)$ is a triangle and so has no edge of a positive slope.

For the converse, suppose that for any Jacobian pair (P,Q), $N(P)$ has no edge of a positive slope. Let us show that such a (P,Q) is an automorphic pair. After a regular linear change of variables we may assume that

$$P(X,Y) = X + F(X,Y), \quad Q(X,Y) = Y + G(X,Y),$$

where $F(X,Y), G(X,Y) \in C[X,Y]$ satisfy $F(X,Y) \equiv 0$ or $O(F) \geq 2$ and $G(X,Y) \equiv 0$ or $O(G) \geq 2$.

We consider the map $f\colon (X,Y) \to (P(X,Y), Q(X,Y))$. Since $\det J(f) \equiv 1$, it follows that f is a local diffeomorphism.

Thus by Hadamard's Theorem it suffices to prove that f has no asymptotic values. If it has asymptotic values then $P(X,Y), Q(X,Y)$ satisfy a double asymptotic identity.

By Newton's algorithm the first power γ in the fractional power series that corresponds to the identity is the negative of a slope of an edge of $W(P)$ and an edge of $W(Q)$. However, since $N(P)$ has no edge of a positive slope it follows that all the slopes of edges of $W(P)$ are strictly less than (-1), except maybe one slope 0.

Similarly, all the slopes of edges of $W(Q)$ are strictly greater than (-1) except maybe for one slope 0.

Since $\gamma \neq 0$ in an asymptotic identity, it follows that (P,Q) cannot satisfy a double asymptotic identity, which completes the proof of the theorem. ∎

An equivalent formulation of Lang's Theorem D is the following

THEOREM 8: *Every Jacobian pair is an automorphic pair iff the Jacobian condition implies that $W(P)$ has no edge of a positive slope.*

Remark 16: Since the edges of slopes different from $0, \infty$ common to $N(P)$ and to $W(P)$ are precisely the edges of a positive slope, we obtain another equivalent condition, namely, the disjointness of the sets of edges of $N(P)$ and of $W(P)$ which do not lie on the axes of the coordinates.

Finally, we mention the following result due to Lang.

THEOREM 9: *Let (P, Q) be a Jacobian pair. Then for every edge of a positive slope of $W(P)$ there coresponds an edge of the same slope in $W(Q)$.*

Proof: The edges of a positive slope in $W(P)$ are precisely the edges of a positive slope in $N(P)$. By Theorem C, $N(P)$ and $N(Q)$ are similar polygons, which implies the conclusion. ∎

6. On the structure of $I(R)$

We recall the definition of $I(R)$ that was given in Section 2 (for the special case $R(X,Y) = (X^{-1}, X + X^2Y)$).

Definition 7: Let $R(X,Y) = (R_1(X,Y), R_2(X,Y))$ be a rational map over R or over C, i.e., $R_j(X,Y) \in K(X,Y)$, $j = 1,2$, where $K = R$ or C.

We shall denote by $I(R)$ the set of all those polynomials $P(X,Y) \in K[X,Y]$ for which

$$(P \circ R)(X,Y) = P(R_1(X,Y), R_2(X,Y)) \in K[X,Y].$$

Thus $I(R)$ consists of all those polynomials that satisfy an asymptotic identity with respect to $R(X,Y)$.

Examples: (1) If $R(X,Y)$ is a polynomial map, then $I(R) = K[X,Y]$.

(2) The counterexample to the real Jacobian conjecture found by S. Pinchuk consists of a real Jacobian pair $(P(X,Y), Q(X,Y))$ (i.e. non-vanishing Jacobian) whose components $P(X,Y), Q(X,Y)$ both belong to $I(X^{-1}, X + X^2Y)$.

(3) As is easy to see, we have

$$I(X^{-1}, X + X^2Y) = K[Y, XY, X^2Y - X]$$

(we shall see that later).

It is rather easy to construct pairs of polynomials in $I(X^{-1}, X + X^2Y)$ over R, say, whose Jacobian is non-negative. For example, if we denote

$$(u_1, u_2, u_3) = (Y, XY, X^2Y - X),$$

then we have

$$J(u_3 - u_1, (u_3 + u_1)(u_2 - 1/2)) = (u_3 + u_1)^2 + 4(u_2 - 1/2)^2,$$

or equivalently

$$J(X^2Y - X - Y, (X^2Y - X + Y)(XY - 1/2)) = (X^2Y - X + Y)^2 + 4(XY - 1/2)^2.$$

By Hadamard's Theorem, any map whose components lie in $I(X^{-1}, X + X^2Y)$ is not injective. The above Jacobian vanishes at only two points, $(\pm 1, \pm 1/2)$, and it is not etale at those points, so that the map is not an example of an etale map which is not a global diffeomorphism as in Pinchuk's construction.

The above example as well as the construction of Pinchuk can be based on the following simple relations among the generators of $I(X^{-1}, X + X^2Y)$. These relations are of two types. One type is algebraic and the second is differential.

ALGEBRAIC RELATIONS.

$$u_1 u_3 = u_2(u_2 - 1).$$

DIFFERENTIAL RELATIONS.

$$J(u_1, u_2) = -u_1, \quad J(u_1, u_3) = 1 - 2u_2, \quad J(u_2, u_3) = -u_3.$$

Later, we shall see how to generalize these to asymptotic identities of the general type that correspond to Y-finite asymptotic values (not merely of degree 2).

Moreover, there will be applications to exotic surfaces. In fact, we shall discuss a new type of exoticity that we shall call etale-exoticity and give examples.

(4) One might attempt, following Pinchuk, to construct a counterexample to the complex Jacobian conjecture whose components are polynomials in $I(X^{-1}, X + X^2Y)$. Indeed, if it were possible to find two polynomials $P(X, Y)$, $Q(X, Y) \in I(X^{-1}, X + X^2Y)$ such that $J(P, Q) \equiv 1$, then this would have been a counterexample to Keller's problem.

According to Moh [9], any such example if it exists is of high degree (≥ 100). However, such an attempt will fail because we shall prove later the following.

THEOREM: *If $P(X, Y), Q(X, Y) \in I(X^{-1}, X + X^2Y)$ then $J(P, Q) \not\equiv 1$.*

In fact, we shall prove a theorem that implies the above as a special case. One key to the proof of this theorem, given in this paper, is the fact that for the special rational map $R(X, Y) = (X^{-1}, X^2Y - X)$ the generators u_1, u_2, u_3 of $I(R)$ can be regarded as homogeneous polynomials in X, Y provided that we assign to (X, Y) the weights $(-1, 1)$. That makes u_1 homogeneous of degree 1, u_2 homogeneous of degree 0, and u_3 homogeneous of degree (-1).

(5) We will give other applications of the study of $I(R)$ to geometry. One example will be the study of certain polynomial vector fields.

All the above justify the study of $I(R)$ and of asymptotic identities. In this section we shall be concerned mainly with algebraical aspects of the structure of $I(R)$. Much of this section will be devoted to the study of the special structure of $I(X^{-1}, X + X^2Y)$ that underlies the construction of Pinchuk, and to other examples.

The geometrical aspects will be discussed in later sections.

We start by pointing out some basic properties of $I(R)$: $I(R)$ is closed with respect to addition and with respect to multiplication; $I(R)$ is 'absorbing' or is an 'ideal' with respect to composition, in the following sense:

$$P(X,Y), Q(X,Y) \in I(R), \quad S(X,Y) \in K[X,Y] \Rightarrow S(P(X,Y), Q(X,Y)) \in I(R).$$

Perhaps a simple but important theorem is the following theorem. It establishes an equivalent formulation to Keller's problem in terms of $I(R)$ and can serve as our starting point.

THEOREM 10: *Keller's problem (the Jacobian conjecture in the complex case) is equivalent to the following statement:*

For any rational map $R(X,Y) = (R_1(X,Y), R_2(X,Y))$ where $R_j(X,Y) \in C(X,Y)$, $j = 1,2$, such that at least one of $R_1(X,Y), R_2(X,Y)$ is not in $C[X,Y]$, there is no Jacobian pair in $I(R)$ (i.e., there is no pair $P(X,Y), Q(X,Y) \in I(R)$ such that $J(P,Q) \in C^*$).

Proof: One direction follows from Hadamard's Theorem:

Let us assume that the Jacobian conjecture holds true. Let $R(X,Y)$ be a rational map as above. Thus $R_1(X,Y)$, say, has a singular point (a pole) at (a,b).

Let $P(X,Y), Q(X,Y) \in I(R)$. Then by definition we have

$$P(R_1(X,Y), R_2(X,Y)) = A(X,Y) \in C[X,Y],$$
$$Q(R_1(X,Y), R_2(X,Y)) = B(X,Y) \in C[X,Y].$$

Hence

$$\lim_{(X,Y)\to(a,b)} P(R_1, R_2) = A(a,b) \quad \text{and} \quad \lim_{(X,Y)\to(a,b)} Q(R_1, R_2) = B(a,b),$$

while $|R_1(X,Y)|$ grows to ∞ as $(X,Y) \to (a,b)$. Thus the map (P,Q) has $(A(a,b), B(a,b))$ as its asymptotic value, so by the assumption on the validity of the Jacobian conjecture (P,Q) cannot be a Jacobian pair.

The opposite direction follows from our theorems on asymptotic identities: Let us assume that the condition holds true. We need to verify the validity of Keller's conjecture. Let $P(X,Y), Q(X,Y) \in C[X,Y]$ satisfy $J(P,Q) \in C^*$. We have to show that the map (P,Q) is a global diffeomorphism. If it is not, then it has asymptotic value (α, β). Hence by our theorems on asymptotic identities there are $R_1(X,Y), R_2(X,Y) \in C(X,Y)$ such that at least one of $R_1(X,Y), R_2(X,Y)$ is not in $C[X,Y]$ (suppose that (a,b) is a singularity of $R_1(X,Y)$) such that (P,Q) satisfy a double asymptotic identity with respect to $R(X,Y) = (R_1(X,Y), R_2(X,Y))$. Hence $P(X,Y), Q(X,Y) \in I(R)$. But this violates the assumption on $I(R)$. ∎

Let us consider a typical double asymptotic identity which is Y-finite (corresponds to the real case, see Section 3):

$$\begin{aligned}
&P(X^{-k}, A_0 + A_1 X + \cdots + A_{N-1}X^{N-1} + YX^N)\\
=&B_0(Y) + B_1(Y)X + \cdots + B_M(Y)X^M,\\
&Q(X^{-k}, A_0 + A_1 X + \cdots + A_{N-1}X^{N-1} + YX^N)\\
=&C_0(Y) + C_1(Y)X + \cdots + C_L(Y)X^L.
\end{aligned}$$

If we denote

$$\begin{aligned}
U &= X^{-k},\\
V &= A_0 + A_1X + \cdots + A_{N-1}X^{N-1} + YX^N,
\end{aligned}$$

then

$$\begin{aligned}
X &= U^{-1/k},\\
Y &= (V - A_0)U^{N/k} - A_1U^{(N-1)/k} - \cdots - A_{N-1}U^{1/k},
\end{aligned}$$

so that we can rewrite the double asymptotic identity as follows:

$$\begin{aligned}
P(U,V) &= \sum_{l=0}^{M} B_l((V - A_0)U^{N/k} - \sum_{j=1}^{N-1} A_j U^{(N-j)/k})U^{-l/k},\\
Q(U,V) &= \sum_{l=0}^{L} C_l((V - A_0)U^{N/k} - \sum_{j=1}^{N-1} A_j U^{(N-j)/k})U^{-l/k}.
\end{aligned}$$

Motivated by the first theorem of this section we would like to show how such a structure of two polynomials should violate the Jacobian condition, i.e., $J(P,Q) \in C^*$. Sometimes that is an easy task.

Example 1: If $A_1 = \cdots = A_{N-1} = 0$ then $J(P,Q)(X, A_0) = 0$.

Proof: This example has, what we shall call later, the property of the shrinking curve:

We have for this case

$$P(X,Y)=\sum_{l=0}^{M} B_l(X^{N/k}(Y-A_0))X^{-l/k},$$

$$Q(X,Y)=\sum_{l=0}^{L} C_l(X^{N/k}(Y-A_0))X^{-l/k}.$$

Hence $P(X,A_0)=\sum_{l=0}^{M} B_l(0)X^{-l/k}\in C[X]$, so that $\alpha=P(X,A_0)\in C$. Similarly $\beta=Q(X,A_0)\in C$. So the map (P,Q) shrinks the curve $\{(X,A_0)\mid X\in C\}$ to the single point (α,β). In particular $J(P,Q)(X,A_0)=0$. ∎

Another example of that type which is X-finite is the following.

Example 2: If $P(X,Y),Q(X,Y)\in I(X,a/X+Y/X^2)$, $a\in C$, then $J(P,Q)(0,Y)=0$.

Proof: Let us denote

$$P(U,V)=\sum a_{ij}U^iV^j,\quad P_-(U,V)=\sum_{0\le j\le i} a_{ij}U^iV^j,$$

$$P_+(U,V)=\sum_{0\le i<j} a_{ij}U^iV^j.$$

Then

$$P_+(X,a/X+Y/X^2)=\sum_{0\le i<j} a_{ij}X^i(a/X+Y/X^2)^j\notin C[X,Y]-\{0\}.$$

So the $P_+(U,V)$ part of $P(U,V)$ must be cancelled out:

$$\begin{aligned}P_-(X,a/X+Y/X^2)&=\sum_{0\le j\le i} a_{ij}X^i(a/X+Y/X^2)^j\\&=\sum_{0\le j\le i} a_{ij}X^{i-j}(a+Y/X)^j\\&=\sum_{0\le j\le i} a_{ij}\sum_{k=0}^{j}\binom{j}{k}a^{j-k}X^{i-j-k}Y^k.\end{aligned}$$

Since $P(U,V)\in I(U,a/U+V/U^2)$, we can take in the inner summation only those values of k for which $0\le i-j-k$ or $k\le i-j$. The other terms will cancel $P_+(X,a/X+Y/X^2)$ exactly. We obtain

$$P(X,a/X+Y/X^2)=\sum_{0\le j\le i} a_{ij}\sum_{k=0}^{\min(j,i-j)} X^{i-j-k}Y^k\binom{j}{k}a^{j-k}.$$

We change variables by

$$U = X, \quad V = a/X + Y/X^2,$$

so

$$X = U, \quad Y = U(UV - a),$$

and hence

$$P(U,V) = \sum_{0 \le j \le i} a_{ij} \sum_{k=0}^{\min(j,i-j)} \binom{j}{k} a^{j-k} U^{i-j} (UV - a)^k.$$

Since $k \le i - j$, we see that $P(U,V)$ is generated by monomials of the type

$$U^l (UV - a)^k \quad \text{where } 0 \le k \le l.$$

It is easy to check that $U^l(UV-a)^k \in I(U, a/U + V/U^2)$, and so we have proved the following

PROPOSITION 5: $I(X, a/X + Y/X^2) = C[X, X(XY - a)]$.

An immediate conclusion is that $P(0,Y), Q(0,Y) \in C$. Hence the map (P,Q) shrinks the curve $\{(0,Y) \mid Y \in C\}$ to a single point and, as a result, $J(P,Q)(0,Y) = 0$, as we had to show. ∎

As opposed to the last two examples we shall now consider the case to which Pinchuk's example belongs.

Example 3: We shall prove in the sequel the following

PROPOSITION 6: $I(X^{-1}, X + X^2Y) = C[Y, XY, X^2Y - X]$.

Proof: Let $P(X,Y) \in I(X^{-1}, X + X^2Y)$. Then by definition we have

$$P(X^{-1}, X + X^2Y) \in C[X,Y].$$

Let us denote

$$P(U,V) = \sum a_{ij} U^i V^j, \quad P_+(U,V) = \sum_{0 \le i \le j} a_{ij} U^i V^j,$$

$$P_-(U,V) = \sum_{0 \le j < i} a_{ij} U^i V^j.$$

Then since we clearly have

$$P_+(X^{-1}, X + X^2Y) = \sum_{0 \le i \le j} a_{ij} X^{-i} (X + X^2Y)^j$$

$$= \sum_{0 \le i \le j} a_{ij} X^{j-i} (1 + XY)^j \in C[X,Y],$$

we deduce that $P_+(X,Y) \in I(X^{-1}, X + X^2Y)$ and, since $P = P_+ + P_-$, we deduce that $P_-(X,Y) \in I(X^{-1}, X + X^2Y)$. We have

$$P_-(X^{-1}, X + X^2Y) = \sum_{0 \le j < i} a_{ij} X^{j-i}(1 + XY)^j$$

$$= \sum_{0 \le j < i} a_{ij} \sum_{k=0}^{j} \binom{j}{k} X^{j+k-i} Y^k,$$

and this last sum should be a polynomial in (X, Y). So in the inner summation we can take only those values of k for which $0 \le j + k - i$, i.e., $i - j \le k$. Also since $k \le j$, the effective values of j must satisfy $i - j \le j$ or $i \le 2j < 2i$. Summarizing all these details, we obtain

$$P_-(X^{-1}, X + X^2Y) = \sum_{0 < i/2 \le j < i} a_{ij} \sum_{k=i-j}^{j} \binom{j}{k} X^{j+k-i} Y^k.$$

As in the previous examples we make the change of variables

$$U = X^{-1}, \quad V = X + X^2Y,$$

thus

$$X = U^{-1}, \quad Y = U^2V - U.$$

By substituting, we obtain

$$P_-(U,V) = \sum_{0 < i/2 \le j < i} a_{ij} \sum_{k=i-j}^{j} \binom{j}{k} U^{-(j+k-i)}(U^2V - U)^k$$

$$= \sum_{0 < i/2 \le j < i} a_{ij} \sum_{k=i-j}^{j} \binom{j}{k} U^{i-j}(UV - 1)^k.$$

All the above shows that $P_+(U,V)$ is generated by U^iV^j where $0 \le i \le j$, and $P_-(U,V)$ is generated by $U^l(UV-1)^k$ where $0 \le l \le k$. Hence both $P_+(U,V)$ and $P_-(U,V)$ are generated by $(V, UV, U^2V - U)$ and, since these generators belong to $I(U^{-1}, U^2V + U)$, we have proved our assertion. ∎

We note that the last proposition implies the following:

$$P(X,Y) \in I(X^{-1}, X + X^2Y) \Rightarrow P_X(X,Y) \in I(X^{-1}, X + X^2Y),$$

To end the discussion of Example 3 we add the following

PROPOSITION 7: *The ring $I(X^{-1}, X+X^2Y)$ does not have the property of the shrinking curve.*

Proof: By the construction of Pinchuk [13] there exist two polynomials $P(X,Y)$, $Q(X,Y) \in I(X^{-1}, X+X^2Y)$ such that

$$J(P,Q)(X,Y) > 0 \quad \forall (X,Y) \in R^2.$$

Alternatively, we could have taken $P(X,Y) = Y$, $Q(X,Y) = X^2Y - X$. ∎

If we review the proofs of the first two propositions of this section we can find some pattern. Let us sketch the method and then demonstrate it in a few more examples.

Let $R(X,Y) = (R_1(X,Y), R_2(X,Y))$ be a rational map. We would like to find the structure of a general $P(X,Y) \in I(R)$.

STEP 1: Write the canonical expansion

$$P(R_1(X,Y), R_2(X,Y)) = \sum a_{ij} X^i Y^j,$$

for a general $P(U,V) \in C[U,V]$.

STEP 2: Assuming that $P(X,Y) \in I(R)$ we conclude that all the monomials that contain negative powers in the expansion of Step 1 must cancel out. This gives some arithmetical relations between i and j.

STEP 3: We change variables (if possible) according to

$$U = R_1(X,Y), \quad V = R_2(X,Y),$$

and the inverse transformation

$$X = S_1(U,V), \quad Y = S_2(X,Y).$$

We point to the fact that in our examples $S_1(U,V), S_2(U,V)$ were rational functions.

STEP 4: We represent $P(U,V)$ in terms of the new variables with the aid of Step 1 and Step 3:

$$P(U,V) = \sum a_{ij} S_1(U,V)^i S_2(U,V)^j,$$

taking into account the arithmetical constraints of Step 2.

STEP 5: We find the set of generators of $I(R)$.

Remark 17: There is no reason to expect that we should get, in the representation of Step 4, $S_1(U,V)^i S_2(U,V)^j \in C[U,V]$, although this was the case in our first examples.

Example 4: Let us discuss the more general Y-finite case with exponent $k=1$, $I(X^{-1}, A_0 + A_1X + \cdots + A_{N-1}X^{N-1} + YX^N)$.

It will be convenient to denote

$$\phi(X) = A_0 + A_1X + \cdots + A_{N-1}X^{N-1}, \quad d = \deg\phi(X).$$

Thus $d \le N-1$. Referring to Step 3 above we shall invert the transformation

$$U = X^{-1}, \quad V = \phi(X) + X^N Y.$$

The result is

$$X = U^{-1}, \quad Y = U^N(V - \phi(U^{-1})) = U^{N-d}(U^dV - U^d\phi(U^{-1})).$$

Clearly $U^d\phi(U^{-1}) \in C[U]$. The generators suggested in Step 5 are combinations of

$$\begin{aligned} S_1(U,V)^i S_2(U,V)^j &= U^{-i}[U^{N-d}(U^dV - U^d\phi(U^{-1}))]^j \\ &= U^{j(N-d)-i}(U^dV - U^d\phi(U^{-1}))^j. \end{aligned}$$

Since $U^dV - U^d\phi(U^{-1}) \in C[U,V]$, the only obstacle can arise because of the factor $U^{j(N-d)-i}$ with negative exponent. Let us carry out Step 2. For that it will be convenient to use the following notation due to Rogosinski [15]:

$$\phi(X)^k = \sum_{l=0}^{dk} a_l^{(k)} X^l.$$

Then

$$\begin{aligned} P(X^{-1}, \phi(X) + X^NY) &= \sum a_{ij} X^{-i}(\phi(X) + X^NY)^j \\ &= \sum a_{ij} X^{-i} \sum_{k=0}^{j} \binom{j}{k} \phi(X)^k X^{N(j-k)} Y^{j-k} \\ &= \sum a_{ij} \sum_{k=0}^{j} \binom{j}{k} Y^{j-k} X^{N(j-k)-i} \sum_{l=0}^{dk} a_l^{(k)} X^l \\ &= \sum a_{ij} \sum_{k=0}^{j} \binom{j}{k} Y^{j-k} \sum_{l=0}^{dk} a_l^{(k)} X^{l+N(j-k)-i}. \end{aligned}$$

Since we assume $P(U,V) \in I(R)$ we can confine ourselves to those indices for which

$$l + N(j-k) - i \geq 0.$$

If we substitute the new variables (U,V) for (X,Y) we obtain

$$\begin{aligned} X^{l+N(j-k)-i}Y^{j-k} &= U^{-[l+N(j-k)-i]}[U^{N-d}(U^dV - U^d\phi(U^{-1}))]^{j-k} \\ &= U^{i-d(j-k)-l}(U^dV - U^d\phi(U^{-1}))^{j-k}. \end{aligned}$$

Let us denote $L = i-d(j-k)-l$, $K = j-k$; then $K \geq 0$ and $L \leq (N-d)(j-k) = (N-d)K$ (by $l + N(j-k) - i \geq 0$). Similarly we can get a lower bound on L.

All the above shows that if $P(U,V) \in I(R)$, then

$$P(U,V) = \sum_{0 \leq K, L \leq (N-d)K} a_{LK} U^L (U^dV - U^d\phi(U^{-1}))^K.$$

Since $P(U,V)$ is a polynomial, one can conclude from this representation the following (we will not need this fact):

PROPOSITION 8:

$$I(X^{-1}, A_1X + A_2X^2 + \cdots + A_{N-1}X^{N-1} + YX^N) =$$
$$C[Y, XY, X^2Y - A_1X, X^3Y - A_1X^2 - A_2X, \ldots, X^NY - A_1X^{N-1} - \cdots - A_{N-1}X].$$

Remark 18: We note that in fact $I(X^{-1}, A_1X + \cdots + A_{N-1}X^{N-1} + YX^N)$ is generated by those polynomials in it which are of Y-degree 1.

Example 5: Let us discuss the case

$$I(X, X^{-1}(A_0 + A_1X + \cdots + A_{N-1}X^{N-1} + YX^N)) \quad \text{where } A_0 \neq 0.$$

As usual we denote $\phi(X) = A_0 + A_1X + \cdots + A_{N-1}X^{N-1}$. We invert the change of variables as in Step 3:

$$U = X, \quad V = X^{-1}(\phi(X) + X^NY).$$

The result is

$$X = U, \quad Y = U^{-N}(UV - \phi(U)).$$

So the generators are of the form

$$U^i[U^{-N}(UV - \phi(U))]^j = U^{i-Nj}(UV - \phi(U))^j.$$

To accomplish Step 2 we again use Rogosinski's notation

$$P(X, X^{-1}(\phi(X) + X^N Y)) = \sum a_{ij} \sum_{k=0}^{j} \binom{j}{k} \sum_{l=0}^{dk} a_l^{(k)} X^{l+N(j-k)+i-j} Y^{j-k}.$$

So for $P(U,V) \in I(U^{-1}, \phi(U) + U^N V)$ we can sum only over those values of indices for which

$$l + N(j-k) + i - j \geq 0.$$

The change of variables gives

$$X^{l+N(j-k)+i-j} Y^{j-k} = U^{l+i-j}(UV - \phi(U))^{j-k}.$$

If we denote $L = l + i - j$, $K = j - k$, then $K \geq 0$ and $L + NK \geq 0$ (by $l + N(j-k) + i - j \geq 0$). As a consequence we obtain the following:

$P(U,V) \in I(U, U^{-1}(\phi(U) + U^N V))$ iff $P(U,V)$ has the form

$$P(U,V) = \sum_{K \geq 0, L+NK \geq 0} a_{LK} U^L (UV - \phi(U))^K,$$

where $\phi(t) = A_0 + \cdots + A_{N-1} t^{N-1}$.

From here we can read the generators.

Remark 19: We recall that the asymptotic identity of the last example is a special case of the identities which we extracted in the complex case using the Newton's algorithm. These are the following:

$$I(X^k, X^{-l}(A_0 + A_1 X + \cdots + A_{N-1} X^{N-1} + Y X^N)), \quad \text{where } k, l \in Z^+.$$

We can always reduce the discussion of these identities as follows: If $P(X,Y) \in I(X^k, X^{-l}(A_0 + \cdots + A_{N-1} X^{N-1} + Y X^N))$, $l < N$, then

$$P(X^k, Y + A_l + A_{l+1} X + \cdots + A_{N-1} X^{N-1-l})$$
$$\in I(X, X^{-l}(A_0 + \cdots + A_{l-1} X^{l-1}) + Y X^{N-l})).$$

We note that if $k = 1$, then the change of variables we used, i.e.,

$$(X,Y) \to (X, Y + (A_l + \ldots + A_{N-1} X^{N-1-l})),$$

is a $C[X,Y]$ automorphism, since it is an elementary map.

The other type of asymptotic identities we had in the complex case was the one originating from

$$I(X^{-k}, X^{-l}(A_0 + \cdots + A_{N-1} X^{N-1} + Y X^N)),$$

where $k, l \in Z^+$.

Example 6: We shall just quote the result that corresponds to this last type of asymptotic identities with $k = l = 1$. The procedure of proving it is similar to the one we used in Example 5 and in previous examples:

$$P(U,V) \in I(U^{-1}, U^{-1}(A_0 + \cdots + A_{N-1}U^{N-1} + VU^N))$$

iff $P(U,V)$ has the following form:

$$P(U,V) = \sum_{k \geq 0, l \leq Nk} a_{lk} U^l (VU^{-1} - \phi(U^{-1}))^k,$$

where $\phi(t) = A_0 + \cdots + A_{N-1}t^{N-1}$.

7. The non-finiteness of maps whose coordinates belong to $I(R)$

In this section we shall give another formulation for the complex Jacobian conjecture. It will use the notion of finiteness of a map. This will be proved with the aid of the theory of asymptotic identities. We start by recalling the following

Definition 8: Let $P(U,V), Q(U,V) \in K[U,V]$ where K is a field. The map $f(U,V) = (P(U,V), Q(U,V))$ is said to be **a finite map** if for any $r(U,V) \in K[U,V]$ there exists an $m > 0$ and $P_j(P(U,V), Q(U,V)) \in K[P(U,V), Q(U,V)]$, $0 \leq j \leq m-1$, so that

$$r(U,V)^m + P_{m-1}r(U,V)^{m-1} + \cdots + P_0 \equiv 0.$$

Let us extend this definition as follows:

Definition 9: Let K be a field and let $R \subseteq K[U,V]$ be a sub-algebra. The sub-algebra R is said to be **finite (over** $K[U,V]$**)** if for any $r(U,V) \in K[U,V]$ there exists an $m > 0$ and $P_j \in R$, $0 \leq j \leq m-1$, so that

$$r^m + P_{m-1}r^{m-1} + \cdots + P_0 \equiv 0.$$

Remark 20: Clearly, the map $f(U,V) = (P(U,V), Q(U,V))$ is a finite map iff $K[P(U,V), Q(U,V)]$ is finite.

With the aid of the above definition we can now state conveniently our results.

THEOREM 11: *Let $R(X,Y) = (R_1(X,Y), R_2(X,Y))$ be a rational map which is not polynomial, over $K = C$ or R. Then $I(R)$ is not finite over $K[U,V]$.*

Proof: Suppose that $R_1(X,Y)$ has a pole at $(X,Y) = (a,b)$. If $I(R)$ were finite, then in particular we could have found an $m > 0$ and $P_j(U,V) \in I(R)$, $0 \leq j \leq m-1$, so that the following identity was true:

$$U^m + P_{m-1}(U,V)U^{m-1} + \cdots + P_0(U,V) \equiv 0.$$

Let us substitute into the identity $(U,V) = R(X,Y)$. Then by $P_j(U,V) \in I(R)$, $0 \leq j \leq m-1$, we have

$$(P_j \circ R)(X,Y) = Q_j(X,Y) \in K[X,Y], \quad 0 \leq j \leq m-1,$$

and so the result of the substitution was

$$R_1(X,Y)^m + Q_{m-1}(X,Y)R_1(X,Y)^{m-1} + \cdots + Q_0(X,Y) \equiv 0.$$

This is clearly impossible, since the left hand side has a singularity at (a,b). ∎

COROLLARY 4: *Let $R(X,Y)$ be a rational map which is not polynomial, over $K = C$ or R. Then any map of the form $f(U,V) = (P(U,V), Q(U,V))$ where $P(U,V), Q(U,V) \in I(R)$ is not a finite map.*

An immediate consequence will be the following reformulation of the Jacobian conjecture (over C).

THEOREM 12: *The Jacobian conjecture over C is equivalent to the following assertion:*

For any Jacobian pair $P(U,V), Q(U,V) \in C[U,V]$ (i.e. $J(P,Q) \in C^*$), the map $f(U,V) = (P(U,V), Q(U,V))$ is a finite map.

Proof: If the Jacobian conjecture is true and if $P(U,V), Q(U,V) \in C[U,V]$ is a Jacobian pair, then $C[P(U,V), Q(U,V)] = C[U,V]$. So that for any $r(U,V) \in C[U,V]$ there exists an $S(U,V) \in C[U,V]$ such that

$$r(U,V) \equiv S(P(U,V), Q(U,V)).$$

Thus the choice $m = 1$, $P_0 = -(S \circ f)$ shows that f is a finite map.

If the Jacobian conjecture is false, then there is a counterexample $f(U,V) = (P(U,V), Q(U,V))$ such that $P(U,V), Q(U,V) \in I(R)$ for some rational but not polynomial $R(X,Y) = (R_1(X,Y), R_2(X,Y))$. By the previous corollary the map f is not a finite map; hence the assertion is also false. ∎

8. Etale exotic surfaces

In this section we shall explore a connection between the theory of asymptotic identities for polynomial maps and exotic surfaces. An exotic manifold M_n of dimension n is a manifold of dimension n which is diffeomorphic to C^n but which is not isomorphic to it.

In the sequel we will find surfaces S that are diffeomorphic to C^2 via a birational map but for which no regular etale map $S \to C^2$ exists. So, in particular, such a surface S is exotic in the usual sense.

For any dimension $n \geq 3$ we shall find such a surface S_n which will be embedded in C^n.

Let us consider the rational map

$$R(X,Y) = (X^{-1}, X^2Y - X).$$

As we know $I(R) = C[V, VU, VU^2 + U]$. Let us define a surface S_3 in C^3 using the following parametrization:

$$X = V, \quad Y = VU, \quad Z = VU^2 + U.$$

Remark 21: S_3 can be regarded also as a surface in R^3 provided that we take the parameters (U, V) to be in R^2.

We should note that S_3 is diffeomorphic to C^2. In fact, the above parametrization is an embedding of C^2 in C^3. To check injectivity we note that if $(V_1, V_1U_1, V_1U_1^2 + U_1) = (V_2, V_2U_2, V_2U_2^2 + U_2)$, then $V_1 = V_2$. If $V_1 \neq 0$, then since $V_1U_1 = V_2U_2$ we get $U_1 = U_2$. If $V_1 = 0$, then since $V_1U_1^2 + U_1 = V_2U_2^2 + U_2$ we get $U_1 = U_2$.

We can invert this birational embedding in two different ways: We always have $X = V$. If $V \neq 0$, then $U = Y/X$ on S_3. Also in that case $Z = Y(Y+1)/X$. If $V = 0$ then $U = Z$. Hence the inverse of the above embedding $\phi(U, V)$ is given by

$$\phi^{-1}(X,Y,Z) = \begin{cases} (Y/X, X), & X \neq 0 \\ (Z, 0), & X = 0 \end{cases}.$$

If $Y \neq -1$ then $U = Z/(Y+1)$, and if $Y = -1$ then $U = -1/X$. Thus we also have

$$\phi^{-1}(X,Y,Z) = \begin{cases} (Z/(Y+1), X), & Y \neq -1 \\ (-1/X, X), & Y = -1 \end{cases}.$$

The surface $XZ = Y(Y+1)$ strictly includes the parametrized surface S_3. The only difference between the two is the straight line $\{(0, -1, Z) \mid Z \in C\}$, which belongs to $XZ = Y(Y+1)$ but not to S_3. S_3 is not affine closed. Its affine closure is the surface $XZ = Y(Y+1)$.

We are interested in the set of all regular functions on S_3. A function $f: S_3 \to C$ is regular if there is a polynomial $P(X,Y,Z) \in C[X,Y,Z]$ such that $P|_{S_3} = f$. A map $F: S_3 \to C^2$ is regular if both of its components are regular functions.

An immediate consequence of the above is that the set of all regular functions on S_3 is exactly

$$C[V, VU, VU^2 + U] = I(X^{-1}, YX^2 - X).$$

Motivated by the theory of asymptotic identities and its relations to the Jacobian conjecture we suspect that the following is true: there is no regular map $F: S_3 \to C^2$ which is a local diffeomorphism. Equivalently, there is no regular etale map $F: S_3 \to C^2$ (just into C^2). Later on we shall prove a theorem will imply that. As previously mentioned this implies that S_3 is an exotic surface.

However, the proof of this last fact is much easier.

THEOREM 13: *The surface S_3 is not isomorphic to C^2. Equivalently, if $P(U,V), Q(U,V) \in C[V, VU, VU^2+U]$ then the pair (P,Q) is not an automorphism of $C[U,V]$.*

Proof 1 (V. Lin): Let us assume that $P(X,Y,Z), Q(X,Y,Z) \in C[X,Y,Z]$ are such that the map $\psi(X,Y,Z) = (P(X,Y,Z), Q(X,Y,Z))$ restricted to S_3 is a global diffeomorphism. Then we can define a map $\varphi: C^2 \to C^2$ as follows: $\varphi = \psi \circ \phi$ where, as before, $\phi(U,V) = (V, VU, VU^2 + U)$. φ is a polynomial automorphism of C^2 since it is polynomial and injective. It follows that φ^{-1} is also polynomial. But $\varphi^{-1} = \phi^{-1} \circ \psi^{-1}$ so that $\phi^{-1} = \varphi^{-1} \circ \psi$ is polynomial. However, as is easy to check, ϕ^{-1} is not polynomial and hence a contradiction. ∎

Proof 2: We consider the birational map $R(X,Y) = (X^{-1}, YX^2 - X)$. Then we have

$$C[V, VU, VU^2 + U] = I(R(X,Y)).$$

So $U \notin C[V, VU, VU^2 + U]$. Recall the absorbing property of $I(R(X,Y))$: If $r(U,V) \in C[U,V]$ and $P(U,V), Q(U,V) \in I(R(X,Y)) = C[V, VU, VU^2 + U]$ then $r(P(U,V), Q(U,V)) \in C[V, VU, VU^2 + U]$.

Let us assume in order to get a contradiction that $P(U,V)$ and $Q(U,V)$ are in $C[V, VU, VU^2 + U]$ and that (P,Q) is an automorphic pair of $C[U,V]$. Then we can find a polynomial $r(U,V) \in C[U,V]$ such that $U = r(P(U,V), Q(U,V))$. By the absorbing property this implies that $U \in C[V, VU, VU^2 + U]$, which cannot be. ∎

In fact the last theorem is true over R also. The proof that we shall give below proves also the case C once more.

THEOREM 14: *The real S_3 cannot be mapped diffeomorphically onto R^2 by a polynomial map. Equivalently, if $P(U,V), Q(U,V) \in R[V, VU, VU^2 + U]$ then the map (P,Q) is not a global diffeomorphism.*

Proof: Let us consider the rational map $R(X,Y) = (X^{-1}, YX^2 - X)$. Let us assume that $P(U,V), Q(U,V) \in R[V, VU, VU^2 + U]$ are such that $F(U,V) = (P(U,V), Q(U,V))$ is a global diffeomorphism. By the Theorem of Hadamard, F cannot have asymptotic values. There are $A(X,Y), B(X,Y) \in R[X,Y]$ such that

$$F(X^{-1}, YX^2 - X) = (A(X,Y), B(X,Y)),$$

for $P(U,V), Q(U,V) \in I(R(X,Y))$. Hence

$$\lim_{X \to 0} F(X^{-1}, YX^2 - X) = (A(0,Y), B(0,Y)) \in R^2.$$

So the curve $\{(A(0,Y), B(0,Y)) \mid Y \in R\}$ consists of, asymptotic values of F, which is a contradiction. ∎

We now return to the question of regular etale maps $F\colon S_3 \to C^2$ and $F\colon S_3 \to R^2$.

THEOREM 15 (S. Pinchuk): *Let S_3 be the real surface; then there are regular etale maps $F\colon S_3 \to R^2$.*

Equivalently, there are polynomials $P(U,V), Q(U,V) \in R[V, VU, VU^2 + U]$ such that the map (P,Q) has a non-vanishing Jacobian.

For, as we remarked previously, the counterexample to the real Jacobian conjecture that was constructed by S. Pinchuk consists of a map both of whose components are in $R[V, VU, VU^2 + U]$.

The complex case has a different answer. However, the proof of that fact is more involved. In fact we shall now prove a more general theorem that will imply the result. The proof is based on an idea of L. Makar-Limanov to whom I am grateful.

This time we consider the rational map

$$R(X,Y) = (X^{-1}, YX^N - X),$$

where $N \geq 2$ is fixed. As we know $I(R) = C[V, VU, VU^2+U, \ldots, VU^N+U^{N-1}]$. Let us define the surface S_{N+1} in C^{N+1} using the following parametrization:

$$X_0 = V, \quad X_1 = VU, \quad X_2 = VU^2 + U, \quad \ldots, \quad X_N = VU^N + U^{N-1}.$$

This is an embedding of C^2 in C^{N+1} and C^2 is diffeomorphic to S_{N+1}; S_{N+1} is not affine closed. Its affine closure is given by

$$X_0X_2 = X_1(X_1+1),$$
$$X_0X_j = X_1X_{j-1}, \quad 3 \leq j \leq N.$$

THEOREM 16: *There are no regular etale maps $F\colon S_{N+1} \to C^2$. Equivalently, if $P(U,V), Q(U,V) \in C[V, VU, VU^2+U, \ldots, VU^N+U^{N-1}]$ then the Jacobian $\partial(P,Q)/\partial(U,V)$ must have a zero.*

Proof: We shall use a weighted grading on $C[U,V]$. The weights are chosen as follows:

$$\deg U = -1, \quad \deg V = 1.$$

Thus $\deg U^iV^j = j - i$ and, for the generators of S_{N+1}, we have the following facts: $V, VU, VU^2+U, \ldots, VU^N+U^{N-1}$ are homogeneous (with respect to the weights),

$$\deg V = 1, \quad \deg(VU) = 0,$$
$$\deg(VU^2+U) = -1, \quad \ldots, \quad \deg(VU^N+U^{N-1}) = -(N-1).$$

Let us denote $T = VU$. We will need to know the structure of homogeneous polynomials with respect to the chosen weights.

If $P_k(U,V) \in C[U,V]$ is homogeneous of degree $k \geq 0$, then there exists a polynomial $q(T) \in C[T]$ such that

$$P_k(U,V) = V^k q(T). \tag{12}$$

To see that we write the following:

$$\begin{aligned} P_k(U,V) &= \sum a_{ij}U^iV^j = \sum_{\deg(U^iV^j)=k} a_{ij}U^iV^j \\ &= \sum_{j-i=k} a_{ij}U^iV^j = \sum a_{i(i+k)}U^iV^{i+k} \\ &= V^k \sum a_{i(i+k)}T^i. \end{aligned}$$

A little more complicated is the structure of homogeneous polynomials of negative degree. Here we shall confine ourselves to $C[V, VU, VU^2+U, \ldots, VU^N+U^{N-1}]$.

If $P_{-k}(U,V) \in C[V, VU, VU^2+U, \ldots, VU^N+U^{N-1}]$ is homogeneous of degree $-k<0$, then the following holds true:

(13)
$$P_{-k}(U,V) = \sum_{j_2+2j_3+\cdots+(N-1)j_N=k} (VU^2+U)^{j_2}\cdots(VU^N+U^{N-1})^{j_N} P_{j_2\cdots j_N}(T),$$

where $P_{j_2\cdots j_N}(T) \in C[T]$.

To see that we write the following:

$$\begin{aligned} P_{-k}(U,V) &= \sum_{i_0-i_2-2i_3-\cdots-(N-1)i_N=-k} a_{i_0\cdots i_N} V^{i_0}(VU)^{i_1}\cdots(VU^N+U^{N-1})^{i_N} \\ &= \sum_{i_0+k=i_2+\cdots+(N-1)i_N} a_{i_0\cdots i_N} V^{i_0}\cdots(VU^N+U^{N-1})^{i_N} \\ &= \sum[(VU^2+U)^{j_2}\cdots(VU^N+U^{N-1})^{j_N}]\times \\ &\quad\times [a_{i_0\cdots i_N} V^{i_0}(VU)^{i_1}(VU^2+U)^{i_2-j_2}\cdots(VU^N+U^{N-1})^{i_N-j_N}], \end{aligned}$$

where in the last sum the indices vary as follows:

$$i_0 = (i_2-j_2)+2(i_3-j_3)+\cdots+(N-1)(i_N-j_N),$$
$$i_2-j_2,\ldots,i_N-j_N \ge 0,$$
$$j_2+2j_3+\cdots+(N-1)j_N = k.$$

Since $V^{i_0}(VU)^{i_1}(VU^2+U)^{i_2-j_2}\cdots(VU^N+U^{N-1})^{i_N-j_N}$ belongs to $C[T]$, by $i_0=(i_2-j_2)+\cdots+(N-1)(i_N-j_N)$ this proves equation (13).

We now consider the Jacobian of a pair of polynomials in

$$C[V, VU, VU^2+U, \ldots, VU^N+U^{N-1}]$$

which are homogeneous of degrees k and $-k$, respectively, for some $k \ge 0$.

In fact, we shall confine ourselves to those polynomials of negative degree that appear in the sum in equation (13).

If $k \ge 0$ and

$$P_k(U,V) = V^k f(T), \quad Q_{-k}(U,V) = (VU^2+U)^{j_2}\cdots(VU^N+U^{N-1})^{j_N} g(T),$$

where $f(T), g(T) \in C[T]$ and $j_2+\cdots+(N-1)j_N = k$, then

(14) $$\partial(P_k, Q_{-k})/\partial(U,V) = -kd/dT\{T^k(T+1)^{j_2+\cdots+j_N} f(T)g(T)\}.$$

The verification of this equation is done by a straightforward computation.

Namely, we have

$$\begin{aligned}
\partial P_k/\partial U &= V^{k+1}f'(T), \quad \partial P_k/\partial V = kV^{k-1}f(T) + V^kUf'(T),\\
\partial Q_{-k}/\partial U =&(VU^2+U)^{j_2-1}\cdots(VU^N+U^{N-1})^{j_N-1}\times\\
&\times\{j_2(VU^3+U^2)\cdots(VU^N+U^{N-1})(2UV+1)+\cdots\\
&\cdots+j_N(VU^2+U)\cdots(VU^{N-1}+U^{N-2})\times\\
&\times(NVU^{N-1}+(N-1)U^{N-2})\}g(T)+\\
&+(VU^2+U)^{j_2}\cdots(VU^N+U^{N-1})^{j_N}Vg'(T),\\
\partial Q_{-k}/\partial V =&(VU^2+U)^{j_2-1}\cdots(VU^N+U^{N-1})^{j_N-1}\times\\
&\times\{j_2(VU^3+U^2)\cdots(VU^N+U^{N-1})U^2+\cdots\\
&\cdots+j_N(VU^2+U)\cdots(VU^{N-1}+U^{N-2})U^N\}g(T)+\\
&+(VU^2+U)^{j_2}\cdots(VU^N+U^{N-1})^{j_N}Ug'(T).
\end{aligned}$$

Now on multiplying and subtracting in

$$\partial(P_k,Q_{-k})/\partial(U,V) = (\partial P_k/\partial U)(\partial Q_{-k}/\partial V) - (\partial P_k/\partial V)(\partial Q_{-k}/\partial U),$$

we find that the coefficient of $f'(T)g'(T)$ is zero while $f'(T)g(T)$ and $f(T)g'(T)$ have the common coefficient

$$-kT^k(T+1)^{j_2+\cdots+j_N},$$

and $f(T)g(T)$ has as its coefficient

$$-kd/dT\{T^k(T+1)^{j_2+\cdots+j_N}\};$$

here as above $k=j_2+\cdots+(N-1)j_N$. In order to prove the theorem we argue by a contradiction. Suppose that

$$P(U,V),Q(U,V)\in C[V,VU,VU^2+U,\ldots,VU^N+U^{N-1}]$$

and that $\partial(P,Q)/\partial(U,V)\equiv 1$. Let us represent P and Q according to our weighted grading

$$P=\sum P_n,\quad Q=\sum Q_m,$$

where P_n and Q_m are homogeneous of degrees n and m, respectively. Then

$$\partial(P,Q)/\partial(U,V)=\sum_{n,m}\partial(P_n,Q_m)/\partial(U,V).$$

However

$$\partial(P_n,Q_m)/\partial(U,V) = (\partial P_n/\partial U)(\partial Q_m/\partial V) - (\partial P_n/\partial V)(\partial Q_m/\partial U),$$

and degree calculations give

$$\deg\{(\partial P_n/\partial U)(\partial Q_m/\partial V)\} = (n+1)+(m-1) = n+m,$$
$$\deg\{(\partial P_n/\partial V)(\partial Q_m/\partial U)\} = (n-1)+(m+1) = n+m.$$

It follows that the homogeneous polynomial $\partial(P_n, Q_m)/\partial(U,V)$ is either identically zero or of degree $n+m$.

Since we assumed that $\sum_{n,m} \partial(P_n, Q_m)/\partial(U,V) \equiv 1$, it follows that

$$\sum_k \partial(P_k, Q_{-k})/\partial(U,V) \equiv 1.$$

Hence by (12), (13) and (14) we obtain an equation of the form

$$\sum_k \left\{ -k \sum_{j_2+\cdots+(N-1)j_N=k} d/dT[T^k(T+1)^{j_2+\cdots+j_N} f(T)g(T)] \right\} \equiv 1,$$

where the polynomials $f(T)$ and $g(T)$ depend on the indices of the summation. Hence we deduce that

$$\sum_k \left\{ -k \sum_{j_2+\cdots+(N-1)j_N=k} T^k(T+1)^{j_2+\cdots+j_N} f(T)g(T) \right\} = T+\lambda,$$

for some $\lambda \in C$.

We note that $T(T+1)$ divides the left hand side of the last equation and so $T(T+1) \mid (T+\lambda)$. This contradiction proves the theorem. ∎

Remark 22: In the case $N=2$ equation (13) becomes

$$P_{-k}(U,V) = (VU^2+U)^k q(T).$$

and equation (14) becomes

$$\partial(P_k, Q-k)/\partial(U,V) = -kd/dT\{[T(T+1)]^k f(T)g(T)\}.$$

As an immediate consequence of the last theorem we deduce that no counterexample to the complex Jacobian conjecture exists of the type found by S. Pinchuk in the real case. More precisely

THEOREM 17: *There is no counterexample to the complex Jacobian conjecture with coordinate polynomials in $C[V, VU, VU^2+U, \ldots, VU^N+U^{N-1}]$, where $N \geq 2$ is fixed.*

The case $N = 2$ of Theorem 16 can be interpreted as a property of

$$\mathrm{SL}_2(C) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in C, ad - bc = 1 \right\}.$$

This follows by an idea of H. Kraft. $\mathrm{SL}_2(C)$ contains the subgroup

$$C^* = \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \mid t \in C^* \right\},$$

which acts on $\mathrm{SL}_2(C)$ by right multiplication

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = \begin{pmatrix} ta & t^{-1}b \\ tc & t^{-1}d \end{pmatrix}.$$

This action has the following 4 invariant functions:

$$ab, ad, bc, cd.$$

We consider the quotient $\mathrm{SL}_2(C)/C^*$, i.e., we define the following equivalence relation on $\mathrm{SL}_2(C)$:

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \sim \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \Leftrightarrow$$
$$\exists A \in C^*, \quad \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} A = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}.$$

THEOREM (H. Kraft): *There is a one to one and onto map between the points of* $\mathrm{SL}_2(C)/C^*$ *and the points of the affine surface* $XZ = Y(Y+1)$ *in* C^3. *This map is defined as follows:*

$$\mathrm{SL}_2(C)/C^* \to \{XZ = Y(Y+1)\},$$
$$\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right] \to (ab, bc, cd).$$

Thus according to Theorem 16 there is no etale map from $\mathrm{SL}_2(C)/C^*$ into C^2.

Remark 23: It is easy to check that a polynomial

$$P(a,b,c,d) = \sum a_{ijkl} a^i b^j c^k d^l \in C[a,b,c,d]$$

is a function on $\mathrm{SL}_2(C)/C^*$ iff $i + k = j + l$.

Let us generalize Theorems 13 and 14 in accordance with our theory of asymptotic identities. Let us consider the rational map

$$R(X,Y) = (X^{-1}, A_1X + \cdots + A_{N-1}X^{N-1} + YX^N), \quad N \geq 2,$$

where at least one of the coefficients $A_1, \ldots, A_{N-1}$ is different from 0. We have

$$I(R) = C[V, VU, VU^2 - A_1U, VU^3 - A_1U^2 - A_2U, \ldots, VU^N - A_1U^{N-1} - \cdots - A_{N-1}U],$$

This motivates our definition for the surface S_R that is induced by $R(X,Y)$ as follows:

$$X_1 = V, \quad X_2 = VU,$$
$$X_3 = VU^2 - A_1U, \quad \ldots, \quad X_{N+1} = VU^N - A_1U^{N-1} - \cdots - A_{N-1}U.$$

Notice that $S_R \subseteq C^{N+1}$.

PROPOSITION 9: *The map* $\phi_R: C^2 \to C^{N+1}$, *defined by*

$$\phi_R(U,V) = (V, VU, VU^2 - A_1U, \ldots, VU^N - A_1U^{N-1} - \cdots - A_{N-1}U),$$

is injective.

Proof: If $\phi_R(U_1, V_1) = \phi_R(U_2, V_2)$ then $V_1 = V_2$. If $V_1 \neq 0$ then, by $U_1V_1 = U_2V_2$, we get $U_1 = U_2$. If $V_1 = 0$, we let j be the smallest index $1 \leq j \leq N-1$, for which $A_j \neq 0$. Then by

$$-A_jU_1 = V_1U_1^{j+1} - A_jU_1 = V_2U_2^{j+1} - A_jU_2 = -A_jU_2,$$

we again obtain $U_1 = U_2$. ∎

Remark 24: In fact ϕ_R above is an embedding.

The surface S_R is not affine closed. The next proposition determines its affine closure.

PROPOSITION 10: *The surface* S_R *is an open subset of the closed affine variety* H_R *which is given by the equations*

$$X_1X_{j+2} = X_2(X_{j+1} - A_j), \quad 1 \leq j \leq N-1.$$

Proof: Clearly both S_R and H_R are 2-dimensional in C^{N+1}. For $1 \le j \le N-1$ we obtain by the definition of S_R

$$X_1 X_{j+2} = V(VU^{j+1} - A_1 U^j - \cdots - A_j U),$$
$$X_2(X_{j+1} - A_j) = VU(VU^j - A_1 U^{j-1} - \cdots - A_{j-1}U - A_j).$$

This shows that $S_R \subseteq H_R$ and clearly S_R is open in H_R. ∎

As before, we are interested in the set of all regular functions on S_R. An immediate consequence of our definitions is that the set of all regular functions on S_R is exactly

$$I(X^{-1}, A_1 X + \cdots + A_{N-1}X^{N-1} + YX^N) =$$
$$C[V, VU, VU^2 - A_1 U, \ldots, VU^N - A_1 U^{N-1} - \cdots - A_{N-1}U].$$

Motivated by the theory of asymptotic values and its relations to the Jacobian conjecture we suspect that there are no regular etale maps F: $S_R \to C^2$. *We do not have a proof of such a result which is very close to a verification of the Jacobian conjecture over C.*

We can, however, easily extend Theorems 13 and 14 using basically the same proofs that were given there.

THEOREM 18: *The surface S_R is not isomorphic to C^2. Equivalently, if $P(U,V)$, $Q(U,V) \in C[V, VU, VU^2 - A_1U, \ldots, VU^N - A_1U^{N-1} - \cdots - A_{N-1}U]$ then the pair (P,Q) is not an automorphism of $C[U,V]$.*

THEOREM 19: *If $A_1, \ldots, A_{N-1} \in R$, then the real S_R cannot be mapped diffeomorphically onto R^2 by a polynomial map. Equivalently, if $P(U,V), Q(U,V) \in R[V, VU, VU^2 - A_1, \ldots, VU^N - A_1U^{N-1} - \cdots - A_{N-1}U]$ then the map (P,Q) is not a global diffeomorphism.*

Thus according to these theorems the surfaces S_R are exotic. However, over C we do not know if these surfaces are etale exotic.

The situation over R is settled as follows:

THEOREM 20 (S. Pinchuk): *If $A_1, \ldots, A_{N-1} \in R$ and S_R is the real surface, then there are regular etale maps F: $S_R \to R^2$.*

Equivalently, there are polynomials

$$P(U,V), Q(U,V) \in R[V, VU, VU^2 - A_1U, \ldots, VU^N - A_1U^{N-1} - \cdots - A_{N-1}U]$$

such that the map (P,Q) has a non-vanishing Jacobian.

Proof: We can assume that $A_1 \neq 0$, for otherwise we make a linear change of variables of the form $(U, V) \to (U + \epsilon, V)$.

Then we can assume that $A_1 = -1$, for otherwise we make the linear change of variables $(U, V) \to (-A_1 U, V)$.

By the result of S. Pinchuk there are polynomials

$$P(U, V), Q(U, V) \in R[V, VU, VU^2 + U]$$

such that (P, Q) has a non-vanishing Jacobian. Since $R[V, VU, VU^2 + U] \subseteq R[V, VU, VU^2 + U, \ldots, VU^N + U^{N-1} - \cdots - A_{N-1}U]$ the result follows. ∎

As a final result which is related to the surfaces S_R let us note the following

PROPOSITION 11: *The following are equivalent:*

(a) *There exists a regular etale map $F: S_R \to C^2$.*

(b) *There exists a regular map $G: C^{N+1} \to S_R$ such that $G\,|_{S_R}$ is etale.*

(c) *There exist $P(X_1, \ldots, X_{N+1}), Q(X_1, \ldots, X_{N+1}) \in C[X_1, \ldots, X_{N+1}]$ such that the map $\phi_R(P, Q)$ is etale when restricted to S_R.*

Proof: Clearly (b) $\Leftrightarrow$ (c).

(a) $\Rightarrow$ (b): We take $G = \phi_R \circ F$.

(b) $\Rightarrow$ (a): Let

$$G(X_1, \ldots, X_{N+1}) = (B_1(X_1, \ldots, X_{N+1}), \ldots, B_{N+1}(X_1, \ldots, X_{N+1}))$$

be such that $G\,|_{S_R}$ is etale. Let j be the smallest index, $1 \leq j \leq N - 1$, for which $A_j \neq 0$. We claim that we can take

$$F(X_1, \ldots, X_{N+1}) =$$
$$(B_{j+2}(X_1, \ldots, X_{N+1})/(B_{j+1}(X_1, \ldots, X_{N+1}) - A_j), B_1(X_1, \ldots, X_{N+1})).$$

By the parametrization of S_R it follows that $G = \phi_R \circ F$, so that the only thing that should be verified is that F is regular. We shall see that

$$B_{j+2}/(B_{j+1} - A_j) \in C[X_1, \ldots, X_{N+1}],$$

namely, by the parametrization $B_1 B_{j+2}/(B_{j+1} - A_j) \in C[X_1, \ldots, X_{N+1}]$ and the denominator does not divide B_1. ∎

Theorem 16 provides a solution to the problem of the non-existence of regular etale maps into C^2 for a special class of surfaces S_R, namely, those which are parametrized as follows:

$$X_1 = V, \quad X_2 = VU, \quad X_3 = VU^2 + \alpha U, \quad \ldots, \quad X_{N+1} = VU^{N+1} + \alpha U^{N-1},$$

where $\alpha \in C^*$.

The next theorem will enlarge the class of surfaces for which we can solve this problem.

THEOREM 21: *Let $N \geq 3$, $\alpha \in C^*$ and $\beta \in C$. Let $S(N+1, \alpha, \beta)$ be the surface in C^{N+1} defined by*

$$X_1 = V, \quad X_2 = VU, \quad X_3 = VU^2 + \alpha U, \quad X_4 = VU^3 + \alpha U^2 + \beta U,$$
$$X_j = VU^{j-1} + \alpha U^{j-2} + \beta U^{j-3} + \sum_{i=j-4}^{1} (\beta^{j-2-i}/\alpha^{j-3-i})U^i, \quad 5 \leq j \leq N+1.$$

Then there are no regular etale maps $F\colon S(N+1, \alpha, \beta) \to C^2$.

Proof: We shall show how to reduce the case of the surface $S(N+1, \alpha, \beta)$ to the case of S_{N+1} which was treated in Theorem 16. This will be done with the aid of the following linear transformation:

$$V = W, \quad U = X + \beta/\alpha.$$

Let us show that with the aid of this transformation we can choose the polynomials

$$W, WX, WX^2 + \alpha X, WX^3 + \alpha X^2, \ldots, WX^N + \alpha X^{N-1}$$

as a set of generators for the algebra of regular functions on $S(N+1, \alpha, \beta)$, thus accomplishing the reduction.

$W = V$ is a generator. Since $VU = W(X + \beta/\alpha) = (WX) + (\beta/\alpha W)$, also WX is a generator:

$$\begin{aligned} VU^2 + \alpha U &= W(X + \beta/\alpha)^2 + \alpha(X + \beta/\alpha) \\ &= (WX^2 + \alpha X) + (2\beta/\alpha WX + \beta^2/\alpha^2 W + \beta), \end{aligned}$$

so $WX^2 + \alpha X$ is a generator:

$$VU^3 + \alpha U^2 + \beta U =$$
$$(WX^3 + \alpha X^2) + \{3\beta/\alpha(WX^2 + \alpha X) + 3\beta^2/\alpha^2 WX + \beta^3/\alpha^3 W + \beta^2/\alpha + \beta\}.$$

So $WX^3 + \alpha X^2$ is a generator.

We continue in this manner, taking advantage of the following identity:

$$W(X + \beta/\alpha)^N + \alpha(X + \beta/\alpha)^{N-1} + \beta(X + \beta/\alpha)^{N-2} + \beta^2/\alpha(X + \beta/\alpha)^{N-3} + \cdots$$
$$\cdots + \beta^{N-2}/\alpha^{N-3}(X + \beta/\alpha)$$

$$= (WX^N + \alpha X^{N-1}) + \sum_{j=1}^{N-2} \binom{N}{j} \beta^j/\alpha^j (WX^{N-j} + \alpha U^{N-j-1})$$
$$+N\beta^{N-1}/\alpha^{N-1}(WX) + \beta^N/\alpha^N W + (N-1)\beta^{N-1}/\alpha^{N-2}.$$

The proof is now completed. ∎

We end this section by indicating that the etale exoticity is in fact a property of many innocent-looking surfaces. The complex sphere is just one such surface.

THEOREM 22: *There are no regular etale maps*

$$F\colon \{(X,Y,Z) \mid X^2+Y^2+Z^2=1\} - \{(Z/2, Z(-1)^{1/2}/2, -1) \mid Z \in C\} \to C^2,$$
$$F\colon \{(X,Y,Z) \mid X^2-Y^2-Z^2=-4\} - \{(Z,-Z,-2) \mid Z \in C\} \to C^2.$$

COROLLARY 5: *There are no etale regular maps from the complex sphere $X^2+Y^2+Z^2=1$ into C^2 or from the one-sheeted hyperboloid $X^2-Y^2-Z^2=-4$ into C^2.*

We now prove the theorem.

Proof: Let us consider the map

$$f(R,S,T) = (1/2(R+T), 1/2(-1)^{1/2}(R-T), 1-(-1)^{1/2}S).$$

Then

$$f^{-1}(X,Y,Z) = (X+(-1)^{1/2}Y, (-1)^{1/2}(Z-1), X-(-1)^{1/2}Y),$$

and hence f is a regular linear transformation of C^3. Let us consider the image of the surface $RT = S(S+2(-1)^{1/2})$ under f. To see what it is, note that

$$X^2+Y^2+Z^2 = (1/2(R+T))^2 + (1/2(-1)^{1/2}(R-T))^2 + (1-(-1)^{1/2}S)^2 = 1,$$

so that the image is contained in the sphere $X^2+Y^2+Z^2=1$. In fact they coincide. This follows immediately by the irreducibility of the sphere, or simply by noting that the image of the sphere under f^{-1} is contained in $RT = S(S+2(-1)^{1/2})$.

The surface $X=V, Y=VU, Z=VU^2+U$ is isomorphic to

$$\{(R,S,T) \mid RT = S(S+2(-1)^{1/2})\} - \{(0,-2(-1)^{1/2},T) \mid T \in C\},$$

and this last surface is mapped by f isomorphically onto

$$\{(X,Y,Z) \mid X^2+Y^2+Z^2=1\} - \{(Z/2, (-1)^{1/2}Z/2, -1) \mid Z \in C\}.$$

So the first part of the theorem follows by Theorem 16 with $N = 2$.

To prove the second part of the theorem we use the same argument and the map

$$g(R,S,T) = (R+T, R-T, 2(S+1)),$$

for which

$$g^{-1}(X,Y,Z) = ((X+Y)/2, Z/2-1, (X-Y)/2);$$

g provides an isomorphism between the surface $RT = S(S+2)$ and the hyperboloid $X^2 - Y^2 - Z^2 = -4$, where the line $\{(0,-2,T) \mid T \in C\}$ is mapped onto $\{(T,-T,-2) \mid T \in C\}$.

That completes the proof of the second part of the theorem. ∎

9. Polynomial vector fields and vector bundles on etale exotic surfaces

In the previous section we introduced the notion of etale exotic surfaces S. These had the following properties:

(a) There is a diffeomorphism ϕ: $C^2 \to S$ which is realized by a birational map ϕ.

(b) There is no regular etale map $S \to C^2$ (into C^2).

Examples of such surfaces are given by the following parametrizations:

$$X_1 = V, \quad X_2 = VU, \quad X_3 = VU^2 + U, \quad \ldots, \quad X_{N+1} = VU^N + U^{N-1}, \quad N \geq 2,$$

which describe surfaces in C^{N+1}. As we observed, the case $N = 2$ implied that the complex sphere $X^2 + Y^2 + Z^2 = 1$ enjoys property (b).

In this section we shall investigate polynomial vector fields on such surfaces S and deduce properties of their tangent bundles.

Let us begin by discussing the case $N = 2$: We take two polynomials $P(X,Y,Z), Q(X,Y,Z) \in C[X,Y,Z]$ and evaluate their Jacobian on the surface S_3 which is defined by

$$X = V, \quad Y = VU, \quad Z = VU^2 + U.$$

We obtain, with the aid of the chain rule, the identity

$$\partial(P(X,Y,Z), Q(X,Y,Z))/\partial(U,V) = -\begin{vmatrix} VU^2+U & -(2VU+1) & V \\ P_X & P_Y & P_Z \\ Q_X & Q_Y & Q_Z \end{vmatrix}.$$

Recall that the affine closure of S_3 is given by $F(X,Y,Z) = XZ - Y(Y+1) = 0$. So on S_3, $F_X = Z = VU^2 + U, F_Y = -(2Y+1) = -(2VU+1)$ and $F_Z = X = V$. Hence the above identity can be written as follows:

$$\partial(P(X,Y,Z), Q(X,Y,Z))/\partial(U,V) = -\partial(F,P,Q)/\partial(X,Y,Z)\,|_{S_3},$$

where all the expressions are evaluated on S_3.

Thus an immediate consequence of Theorem 16 is that for any two polynomials $P(X,Y,Z), Q(X,Y,Z) \in C[X,Y,Z]$ the Jacobian $\partial(F,P,Q)/\partial(X,Y,Z)$ must have zeros on the surface S_3, where here, as above,

$$F(X,Y,Z) = XZ - Y(Y+1).$$

To interpret that geometrically we note that

$$\partial(F,P,Q)/\partial(X,Y,Z) = \nabla F \cdot (\nabla P \times \nabla Q),$$

where we use the standard calculus notations ∇ for gradient, $\cdot$ for scalar product and $\times$ for vector product. Hence this Jacobian vanishes exactly when the vectors ∇F and $\nabla P \times \nabla Q$ are perpendicular.

However, on S_3, ∇F is a normal vector to the surface (always different from 0). Thus we can restate the above conclusion as follows :

For any two polynomials $P(X,Y,Z), Q(X,Y,Z) \in C[X,Y,Z]$ the vector field $\nabla P \times \nabla Q$ on S_3 must contain vectors which lie in the tangent plane to S_3 (at the point of evaluation).

Remark 25: As is easy to check, twice continuously differentiable vector fields $\vee$ of the form $\nabla P \times \nabla Q$ are characterized by the equation $\mathrm{Div} \vee = 0$.

Another way to state the above is to say that the family of planes (maybe degenerated) that are spanned by ∇P and ∇Q on S_3 must contain planes perpendicular to S_3.

Yet we look for a formulation of that which is free of differentiations. We recall that ∇P evaluated on the surface $P(X,Y,Z) \equiv \mathrm{Const}$ is a perpendicular vector to that surface (it might be 0). Thus we arrive at the following statement:

For any two polynomials $P(X,Y,Z), Q(X,Y,Z) \in C[X,Y,Z]$ there are two level surfaces $P(X,Y,Z) = C_1$ and $P(X,Y,Z) = C_2$ such that these two and S_3 have a common point through which a straight line passes that belongs to the 3 tangent planes of these surfaces at this point of intersection.

The above is the core of the results in this section. We now state these formally in a more general setting that we have.

PROPOSITION 12: *Let S_R be the surface in C^{N+1}, $N \geq 2$, which is parametrized by*

$$X_1 = V, \quad X_2 = VU,$$
$$X_3 = VU^2 - A_1U, \quad \ldots, \quad X_{N+1} = VU^N - A_1U^{N-1} - \cdots - A_{N-1}U.$$

Let $F_j(X_1, \ldots, X_{N+1}) = X_1X_{j+2} - X_2(X_{j+1} - A_j)$, $1 \leq j \leq N-1$. Then for any pair of polynomials $P(X_1, \ldots, X_{N+1}), Q(X_1, \ldots, X_{N+1}) \in C[X_1, \ldots, X_{N+1}]$ we have the identity

$$\partial(F_1, \ldots, F_{N-1}, P, Q)/\partial(X_1, \ldots, X_{N+1}) =$$
$$V^{N-2}\partial(P(X_1, \ldots, X_{N+1}), Q(X_1, \ldots, X_{N+1}))/\partial(U, V),$$

where everything is evaluated on S_R.

This proposition is proved using the fact that the affine closure of S_R is given by $F_j = 0$, $1 \leq j \leq N-1$ and by the chain rule.

THEOREM 23: *Let S_{N+1} be the surface in C^{N+1}, $N \geq 2$, which is parametrized by*

$$X_1 = V, \quad X_2 = VU, X_3 = VU^2 + U, \quad \ldots, \quad X_{N+1} = VU^N + U^{N-1}.$$

Let $F_1(X_1, \ldots, X_{N+1}) = X_1X_3 - X_2(X_2 + 1)$, $F_j(X_1, \ldots, X_{N+1}) = X_1X_{j+2} - X_2X_{j+1}$, $2 \leq j \leq N-1$.

Then for any pair of polynomials $P(X_1, \ldots, X_{N+1}), Q(X_1, \ldots, X_{N+1}) \in C[X_1, \ldots, X_{N+1}]$ the Jacobian $\partial(F_1, \ldots, F_{N-1}, P, Q)/\partial(X_1, \ldots, X_{N+1})$ has zero on S_{N+1}.

Proof: This follows from the previous proposition and from Theorem 16. ∎

As a consequence of this theorem we get

THEOREM 24: *For any pair of polynomials $P(X_1, \ldots, X_{N+1}), Q(X_1, \ldots, X_{N+1}) \in C[X_1, \ldots, X_{N+1}]$ there is a point on S_{N+1} at which both of the vector fields given by $\vee_P = (P_{X_1}, \ldots, P_{X_{N+1}})$, $\vee_Q = (Q_{X_1}, \ldots, Q_{X_{N+1}})$ are perpendicular to S_{N+1}.*

An equivalent way of saying that is

THEOREM 25: *For any pair of polynomials $P(X_1, \ldots, X_{N+1}), Q(X_1, \ldots, X_{N+1}) \in C[X_1, \ldots, X_{N+1}]$ there exist two level hypersurfaces $P(X_1, \ldots, X_{N+1}) = C_1$, $Q(X_1, \ldots, X_{N+1}) = C_2$ such that these two and S_{N+1} have a common point, and a straight line through it that is tangent to all the 3 at this point of intersection.*

We end this section by pointing out the possibility of existence of an algebraic version of the Gauss Theorem or the Divergence Theorem. We do not have a conjecture as to what that version should be. Motivated by the above theorems we write the following

Definition 10: The C^1 surface $S(X,Y,Z)=0$ is said to have the property τ with respect to the ring of functions $A \subseteq C^1$ if for any pair $P(X,Y,Z), Q(X,Y,Z) \in A$ there exists a point (X_0,Y_0,Z_0) satisfying

$$S(X_0,Y_0,Z_0) = \partial(S,P,Q)/\partial(X,Y,Z)(X_0,Y_0,Z_0) = 0.$$

Remark 26: (a) If $S(X,Y,Z)=0$ is a polynomial and it has the property τ with respect to $A=C[X,Y,Z]$, then $S(X,Y,Z)$ does not have Jacobian mates.

(b) If $S(X,Y,Z)=0$ has a singular point, then it has the property τ.

(c) The surface $XZ-Y(Y+1)=0$ has the property τ with respect to $A = C[X,Y,Z]$.

It is rather clear how to extend the definition of the property τ to surfaces that are embedded in C^{N+1}, $N \geq 3$.

When one tries to show that a surface S enjoys the property τ with respect to $A=C^2$, one naturally relies on techniques of vector analysis. We shall demonstrate that over R^3.

PROPOSITION 13: *A C^1 closed surface over R (in R^3) has the property τ with respect to $A=C^2(R^3)$.*

Proof: Let S be a C^1 closed surface in R^3 that bounds the domain V. Suppose that this surface does not have the property τ with respect to A. Then there exists a pair $P(X,Y,Z), Q(X,Y,Z) \in C^2(R^3)$ such that the vector field

$$\vee(X,Y,Z) = \nabla P \times \nabla Q \mid_S$$

on S is composed only of vectors for which the angle α between them and the unit normal $n(X,Y,Z)$ to S satisfy $0 \leq \alpha < \pi/2$.

Hence we have

$$\int\!\!\int_S \vee(X,Y,Z)\cdot n(X,Y,Z)dS > 0,$$

where we integrate on the surface S.

We can use the Divergence Theorem in our case. Thus

$$\int\!\!\int\!\!\int_V \mathrm{Div}\vee(X,Y,Z)dXdYdZ = \int\!\!\int_S \vee(X,Y,Z)\cdot n(X,Y,Z)dS.$$

Hence

$$\int\int\int_V \operatorname{Div} \vee(X,Y,Z)dX dY dZ > 0.$$

On the other hand, since $\vee(X, Y, Z) = \nabla P \times \nabla Q$ it follows that

$$\operatorname{Div} \vee(X,Y,Z) \equiv 0,$$

and so we arrive at a contradiction, which proves the theorem. ∎

Such a chain of arguments relies heavily on a theorem such as the Divergence Theorem. These Stokes-type theorems usually require compactness of the surface (or at least a good control on the growth of the vector fields towards infinity). However, our surfaces S_R are certainly not compact. On the other hand, we are interested in the property τ only with respect to the ring of polynomials and so maybe one can prove an algebraic version of the Stokes Theorem that will handle this situation.

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